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Master integrals for massless two-loop vertex diagrams with three offshell legs

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Abstract: We compute the master integrals for massless two-loop vertex graphs with three off-shell legs. These master integrals are relevant for the QCD corrections to $H \to V^*V^*$ (where $V = W, Z$) and for two-loop studies of the triple gluon (and quark-gluon) vertex. We employ the differential equation technique to provide series expansions in ϵ for the various master integrals. The results are analytic and contain a new class of two-dimensional harmonic polylogarithms.

Keywords: QCD, NLO Computations, Jets.

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Contents

1. Introduction

It is well known that many of the loop integrals that appear in Feynman diagram calculations can be expressed in terms of hypergeometric functions with parameters that depend on the number of space time dimensions d and a number of kinematic scales. However, expressing these hypergeometric functions as expansions in $\epsilon = (4-d)/2$ is rather non-trivial. In general the coefficients involve polylogarithms, both of the Nielsen [1] and harmonic [2, 3] varieties, and often new polylogarithms need to be introduced.

Integration by parts [4, 5] (IBP) and Lorentz invariance [6] (LI) identities are crucial in reducing [7] the number of master integrals (MI) that actually need to be evaluated and several powerful tools have been established to deal with the problem. Often, these methods rely on the link between the hypergeometric functions that yield (nested) sums and their integral representations that yield polylogarithms. Two of the most powerful analytic methods are the Mellin-Barnes technique [8] and the differential equations approach [9]. Both have been used extensively to provide expansions in ϵ for two-loop box graphs with massless internal propagators when all the external legs are on-shell [10]–[14] and when one of the external legs is off-shell [15, 16]. All of these expansions have been checked by Binoth and Heinrich's numerical program [17] for evaluating loop integrals. Once analytic expansions for the two-loop master integrals were known, they were rapidly exploited in the calculation of the amplitudes for physical scattering processes.

The two-loop helicity amplitudes have been evaluated for the gluon-gluon [18, 19], quark-gluon [20, 21] and quark-quark [22] processes and have confirmed the earlier "squared" matrix elements [23]–[26]. Similarly, amplitudes for the phenomenologically important $gg \to \gamma\gamma$ [27] and $q\bar{q} \to \gamma\gamma$ [28] processes as well as $\gamma\gamma \to \gamma\gamma$ [29, 30] and (massless) Bhabha scattering [31] have also been computed. The processes with one off-shell leg include $e^+e^- \to 3$ jets [32]–[34] which is crucial in making a precise determination of the strong coupling constant at the NLC. Progress is also being made in calculating the two-loop QED corrections to Bhabha scattering which is of crucial importance in determining the luminosity at the NLC (see the nice review of the current status in ref. [35]). Here there are 33 double box graphs to evaluate of which seven have been studied [35]–[38]. Analytic expressions for the associated vertex graphs are also known [39, 35] and have been employed to calculate the QED [40] and QCD [41] corrections to the massive fermion form factor.

In this paper, we take a first step towards calculating massless two-loop $2 \to 2$ scattering amplitudes with two off-shell legs. These processes include the NNLO QCD corrections to $q\bar{q} \to V^*V^*$ (where $V = W,Z$) and the NLO corrections to $gg \to V^*V^*$. Here we indicate that the gauge bosons are off shell to account for resonance effects in the decay of the gauge boson. As in the on-shell and single off-shell cases, the MI include planar and non-planar box graphs as well as vertex and self energies. Altogether there are 11 planar box and 3 non-planar box master topologies, some of which require 2 or more MI to be computed. Our goal in this paper is much more modest. As a first step, we restrict ourselves to the planar and non-planar vertex graphs that form a simpler subset of the necessary MI.

The MI presented here are ingredients for several interesting two-loop processes in their own right. In Higgs physics, the $H \to V^*V^*$ decay receives QCD corrections when the Higgs couples to gluons (via a heavy top quark loop) which then couple to the weak bosons via a massless quark loop. This may be relevant for Higgs searches in the mass regions where the Higgs decays into two off-shell W bosons. In pure QCD, one can evaluate the two-loop triple gluon and quark-gluon two-loop vertices with massless quarks in a covariant gauge (as well as the gluon-ghost vertex).[1] This is a useful input for Schwinger-Dyson studies of confinement as well as exploring how the Ward-Slavnov-Taylor identities generalise to the off-shell case.

[1]Note that Davydychev and Osland have studied the two-loop case where only one of the legs is off-shell [42]

As in refs. [15, 16, 39, 43, 44, 45, 35], we employ the differential equation technique to evaluate the MI. By building up from the simplest MI, we find differential equations in the external scales for new MI where the inhomogeneous terms involve known MI. Together with appropriate boundary conditions, the integral can then be systematically solved order by order in ϵ. At each order we encounter one-dimensional integrals over the terms in the result for one order lower. These integrals yield polylogarithms and, because of the specific kinematics of the vertex graph with three off-shell legs, we find it necessary to extend the set of two-dimensional harmonic polylogarithms (2-d HPL) to include quadratic factors in the denominator (see also ref. [45]). Our results are therefore presented in terms of the extended set of 2-d HPL's.

Our paper is organised as follows. In section 2, we give a brief summary of the differential equation method for solving MI while in section 2. We define our notation and kinematics in section 3 before we discuss harmonic polylogarithms (HPL) and the extended set of 2-d HPL necessary for these MI in section 4. In any two-loop process, some of the graphs are factorisable and are given as products of one-loop graphs. Therefore, we list the one-loop MI in section 5. We note that expressions to all orders in ϵ for the one-loop triangle with off-shell legs have been provided in terms of log-sine functions by Davydychev [47]. The generalised log-sine functions can be directly related to Nielsen polylogarithms [48] and the all-order epsilon-expansion of one-loop massless vertex diagrams with three off-shell legs is given in terms of Nielsen polylogarithms in ref. [48]. However, here we convert these results into 2-d HPL so that they can simply be combined with the other two-loop MI. Expressions for the two-loop MI are listed in section 6, ordered by the number of propagators. Finally, our results are summarized in section 7.

2. Using differential equations to calculate master integrals

2.1 The differential equations

As discussed in Section 1, we use the differential equation (DE) method first suggested in [9] to evaluate the MI's. This method was expanded in detail in [46], and has since been used to calculate many two-loop MI, including those with multiple off-shell legs or internal masses [15, 16, 39, 43, 44, 45, 35]. Here we present only a brief outline of this method, and for further details in the use of this method to find the ϵ-expansion of master integrals we refer the reader to refs. [15, 39].

The main idea of this method is to derive differential equations in external invariants for the master integrals. These equations are then solved using suitable boundary conditions to fix the constants of integration.

The differential equations are obtained by differentiating the MI with respect to the external momenta. Via the chain rule, linear combinations of these differentials are used to find the first order differential equations in the external invariants. These equations involve tensor loop integrals with additional powers of propagators. However, the IBP and LI identities can be used to simplify the equations[2] so that the differential equation is expressed only in terms of the MI itself and combinations of simpler topologies (i.e. integrals with less denominators). For this reason it is sensible to apply a 'bottom-up' approach, working from simpler to more complicated topologies, so that the MI is the only unknown in the differential equation. In other words, the inhomogeneous part of the differential equation is known (or at least the relevant terms in the ϵ-expansion are known). For some topologies there is more than one MI leading to coupled differential equations. However as there is some freedom in the choice of which two-loop graphs to use as the master integrals, it simplifies the calculation to choose the master integrals such that they have different leading powers of ϵ. In this case, the system of differential equations decouples on expansion in ϵ (see, for example, [39]) .

The differential equations are exact in the space-time dimension d, and can be solved as follows. Consider the inhomogeneous differential equation for the MI F with respect to the external scale x,

$$\frac{d}{dx}F(x) = A(x)F(x) + B(x)\,. \tag{2.1}$$

If $H(x)$ is a solution of the homogeneous equation

$$\frac{d}{dx}H(x) = A(x)H(x)\,, \tag{2.2}$$

then the full solution is given by

$$F(x) = H(x)\left(\int^x H^{-1}(x')B(x')dx' + C\right), \tag{2.3}$$

where the constant C has to be fixed from the boundary conditions. These solutions are generally combinations of hypergeometric functions which are difficult to expand in powers of ϵ. Thus to find the ϵ-expansion of the MI we must systematically expand each master integral F, and all d-dependent terms of the differential equation in powers of ϵ

$$F = \sum_{i=-m}^{n} f^i\epsilon^i\,, \qquad A = \sum_{i=0}^{n+m} a^i\epsilon^i\,, \qquad B = \sum_{i=-m}^{n} b^i\epsilon^i\,, \tag{2.4}$$

where $-m$ is the lowest power of ϵ in the expansion and n is the highest power of ϵ needed. It is assumed that the a^i and b^i are already known. Each coefficient f^i satisfies the differential equation given by,

$$\frac{d}{dx}f^i(x) = \sum_{j=0}^{m-i} a^j(x)f^{i-j}(x) + b^i(x)\,. \tag{2.5}$$

It can be seen that the homogeneous part of all the equations generated by the ϵ-expansion is simply the homogeneous solution H evaluated at $d=4$

$$h(x) = H(x)|_{d=4} \tag{2.6}$$

[2]To achieve this we have made extensive use of the Laporta algorithm [7]. We have coded our own version using FORM [49] and checked it against the Automatic Integral Reduction package AIR [50].

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and so the solution is given by

$$f^i(x) = h(x)\left(\int^x h^{-1}(x')\left(\sum_{j=1}^{m-i} a^j(x')f^{i-j}(x') + b^i(x')\right)dx' + c^i\right), \tag{2.7}$$

where the constants c^i have to be fixed from the boundary conditions at each order in ϵ. Note that in general each coefficient f^i will depend on f^{i-1} so we solve the system of equations order by order, using repeated integration of the lower order results. It is for this reason that we require that $A(x)$ has no poles in ϵ as then f^i would depend on f^{i+1} and the bottom-up approach would not be valid.

2.2 The boundary conditions

In general, the lowest order coefficient in ϵ is determined solely by the boundary conditions. The boundary conditions are either obtained from the differential equation or from the master integral itself. To obtain limits from the differential equation it is necessary to examine the singular points in the coefficients of the differential equation. For example, if eq. (2.1) were to take the form

$$\frac{d}{dx}F(x) = \frac{1}{x-a}F(x) + \frac{1}{x-a}B(x) \tag{2.8}$$

then we could multiply the whole equation by $(x-a)$ and let $x \to a$, then we have

$$0 = F(x)|_{x=a} + B(x)|_{x=a} \tag{2.9}$$

giving the boundary condition on $F(x)$. To obtain boundary conditions from the integral itself, we can use limits where the ϵ expansion is known, for example where an offshell leg becomes massless. In both methods care has to be taken that the integral has a smooth limit at the chosen point so as not to miss or introduce hidden singularities.

If $H(x)$ is divergent at the boundary then the constant C is already determined by the necessary condition,

$$\int^x H^{-1}(x')B(x')dx' + C = 0\Big|_{x=a}, \tag{2.10}$$

which can be fulfilled by choosing the boundary point as the lower integration limit. The solution of the differential equation is then given by

$$F(x) = H(x)\tilde{F}(x) = H(x)\left(\int_a^x H^{-1}(x')B(x')dx'\right). \tag{2.11}$$

It can be easily shown that this function satisfies the boundary condition.

3. Kinematics

We consider the vertex graph with three off-shell legs such that,

$$0 \to p_1 + p_2 + p_3\,. \tag{3.1}$$

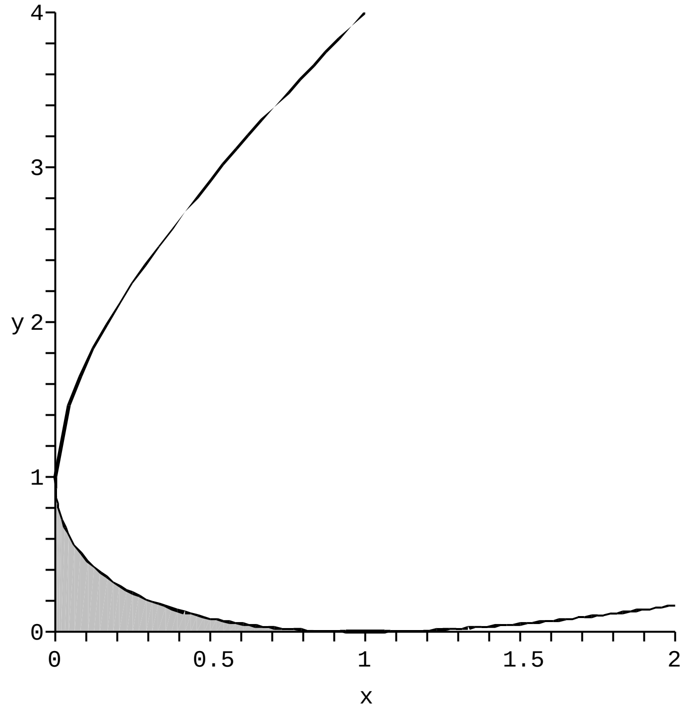


Figure 1: The phase space for the vertex graph with three off-shell legs. The shaded region corresponds to the case where $p_3^2 > p_1^2,\ p_2^2$. The solid line marks the boundary where $\lambda = 0$.

The three kinematic scales are p_i^2. It is convenient to use the dimensionless variables

$$x = \frac{p_1^2}{p_3^2}\,, \qquad y = \frac{p_2^2}{p_3^2} \tag{3.2}$$

together with the determinant of the 2×2 Gram matrix,

$$\lambda \equiv \lambda(x, y) = \sqrt{(1 - x - y)^2 - 4xy} = \sqrt{(x - x_0)(x - x_1)} \tag{3.3}$$

with

$$x_0 = (1 - \sqrt{y})^2\,, \qquad x_1 = (1 + \sqrt{y})^2\,.$$

In the following the dependence of λ, x_0 and x_1 on x and y will be implicitly understood.

In this paper we choose the kinematical configuration to suit the case of a heavy/offshell particle with momentum p_3 decaying into two lighter particles with momenta p_1, p_2. In this case the kinematically accessible region is given by the inequality,

$$\sqrt{p_1^2} + \sqrt{p_2^2} \leq \sqrt{p_3^2}\,, \tag{3.4}$$

or after introduction of the dimensionless parameters x and y

$$\sqrt{x} + \sqrt{y} \leq 1\,. \tag{3.5}$$

The allowed parameter space is depicted by the shaded region in figure 1. Other regions are simply obtained by relabelling.

4. Harmonic polylogarithms

Solving the differential equations by repeated integration immediately suggests that the results be given in terms of Harmonic Polylogarithms (HPL's), whose properties are defined by repeated integration. The HPL's were first introduced in [2] as extensions of Nielsens polylogarithms, and later extended to 2-dimensions by [3]. In this section we briefly review the properties of one-dimensional HPL's (1-d HPL) before detailing the extension of the 2-d HPL's necessary to describe the vertex graph with three off-shell legs.

4.1 1-d HPL's

The weight, w, of a one-dimensional HPL, $H(\vec{b};x)$, is the number of dimensions of the vector of parameters $\vec{b}$. This vector, along with the argument x, fully describes the HPL.

4.1.1 1-d HPL's with $w = 1$

The weight-1 HPL's are defined as follows,

$$H(a;x) = \begin{cases} \int_0^x f(a,x')dx', & a \in \{1,-1\} \\ \int_1^x f(a,x')dx', & a = 0\,, \end{cases} \tag{4.1}$$

where,

$$f(1;x) \equiv \frac{1}{1-x}\,, \tag{4.2}$$

$$f(0;x) \equiv \frac{1}{x}\,, \tag{4.3}$$

$$f(-1;x) \equiv \frac{1}{1+x}\,. \tag{4.4}$$

Note that $H(0;x)$ is defined differently to avoid the logarithmic singularity at $x = 0$. Thus we have

$$H(1;x) \equiv -\ln(1-x)\,, \tag{4.5}$$

$$H(0;x) \equiv \ln x\,, \tag{4.6}$$

$$H(-1;x) \equiv \ln(1+x)\,, \tag{4.7}$$

and

$$\frac{\partial}{\partial x}H(a;x) = f(a;x) \qquad \text{with} \qquad a \in \{+1,0,-1\}. \tag{4.8}$$

4.1.2 1-d HPL's with $w > 1$

The higher weight HPL's are recursively defined by,

$$H(a,\vec{b};x) \equiv \int_0^x dx' f(a;x')H(\vec{b};x)\,, \tag{4.9}$$

$$H(0,\dots,0;x) \equiv \frac{1}{w!}\ln^w x\,. \tag{4.10}$$

Note that only the HPL's with weight vectors comprising only 0's are defined differently, and are integrated between 1 and x to avoid logarithmic singularities. All others involve integration from 0 to x. Under differentiation, the weight is reduced by unity,

$$\frac{\partial}{\partial x} H(a, \vec{b}\,; x) = f(a; x) H(\vec{b}\,; x)\,. \tag{4.11}$$

We also find the following relation useful:

$$H^n(0; x) = n! H(\underbrace{0, \ldots, 0}_{\text{n zeros}}; x) \tag{4.12}$$

4.2 Extension of the 2-d HPL's

The 2-d HPL's were introduced in [3] as the logical extension of the 1-d HPL's. The common extension is the linear basis

$$f(a, x) = \frac{1}{a - x}\,, \tag{4.13}$$

$$f(-a, x) = \frac{1}{a + x}\,. \tag{4.14}$$

However, while it is possible to use this basis to evaluate the integrals under investigation here, a more natural extension involves the square roots that are generated by eq. (3.3). To this end, we extend the 2-d basis by the (quadratic) functions,

$$f(\lambda, x) = \frac{1}{\lambda}\,, \tag{4.15}$$

$$f(x\lambda, x) = \frac{1}{x\lambda}\,, \tag{4.16}$$

$$f(x_0, x) = -\frac{1}{x - x_0}\,, \tag{4.17}$$

$$f(x_1, x) = -\frac{1}{x - x_1}\,. \tag{4.18}$$

These functions are two dimensional, with explicit dependence on x and the dependence on y coming from $x_0(y)$ and $x_1(y)$. The functions are chosen to be positive in the region,

$$0 < x < 1\,, \qquad 0 < y < 1\,, \qquad 0 < \lambda^2\,,$$

which immediately implies

$$0 < x < x_0 = (1 - \sqrt{y})^2 < 1\,, \qquad 0 < y < y_0 = (1 - \sqrt{x})^2 < 1\,, \qquad \sqrt{x} + \sqrt{y} < 1\,, \tag{4.19}$$

and which corresponds to the kinetically allowed region depicted in figure 1.[3]

[3] It is also possible to define a linear basis by making the Euler transformation,

$$\lambda = 2t + x \quad \Rightarrow \quad x = -\frac{(t-a)(t+a)}{(t-b)}$$

with

$$a = -\frac{1-y}{2}\,, \qquad b = -\frac{1+y}{2}$$

such that

$$\frac{dx}{\lambda} \Rightarrow -\frac{dt}{t-b}\,, \qquad \frac{dx}{\lambda^2} \Rightarrow -\frac{dt}{(t-a_1)(t-a_2)}$$

with

$$a_1 = -\frac{1}{2}(1+\sqrt{y})^2\,, \qquad a_2 = -\frac{1}{2}(1-\sqrt{y})^2\,.$$

We define the extended harmonic polylogarithms in the following way,

$$G(a,\vec{w};x,y) = \int_{x_0}^{x} f(a,x')G(\vec{w};x',y)dx' \,, \tag{4.20}$$

and where the dependence on y is made explicit. Note that the lower boundary is $x = x_0$. The only exception from this definition is

$$G(x_0,\ldots,x_0;x,y) = \int_0^x f(x_0,x')G(x_0,\ldots,x_0;x',y)dx' \,. \tag{4.21}$$

However, HPL's of this form do not appear in the results presented in this paper.

The choice of the integration limits is governed by the kinematic boundaries. To be able to evaluate the extended HPL's for $x \to x_0$, the condition $\lim_{x\to x_0} G(\vec{w};x,y) = 0$ has to be fulfilled. This ensures that

$$\int^x f(x_0,x')G(\vec{v};x',y)dx' \tag{4.22}$$

is finite in the limit $x \to x_0$.

Note that HPL's of the form $G(0,\vec{w};x,y)$ and $G(x\lambda,\vec{w};x,y)$ are divergent in the limit that $x \to 0$. This reflects the fact that taking the massless limit and making the ϵ-expansion does not necessarily commute. In the cases where the $x \to 0$ limit is smooth, we find that the HPL's appear in the combination,

$$\Delta G(0,\vec{w};x,y) \equiv G(0,\vec{w};x,y) - (1-y)G(x\lambda,\vec{w};x,y) \tag{4.23}$$

which is finite as $x \to 0$.

For the generalised HPL's of weight 1 we find,

$$G(0;x,y) = \log\left(\frac{x}{x_0}\right), \tag{4.24}$$

$$G(\lambda;x,y) = \log\left(\frac{1-x+y-\lambda}{2\sqrt{y}}\right), \tag{4.25}$$

$$G(x\lambda;x,y) = \frac{1}{1-y}\log\left(\frac{(1-y)(1+x-y-\lambda)-2x}{2x\sqrt{y}}\right), \tag{4.26}$$

$$G(x_0;x,y) = -\log\left(\frac{x_0-x}{x_0}\right), \tag{4.27}$$

$$G(x_1;x,y) = -\log\left(\frac{x_1-x}{x_1-x_0}\right). \tag{4.28}$$

In the course of solving the differential equations for the MI, we also find integrands of the form,

$$\frac{1}{\lambda^2}, \ \frac{1}{(x-x_{0,1})\lambda} \approx (x-x_0)^\alpha (x-x_1)^\beta \,.$$

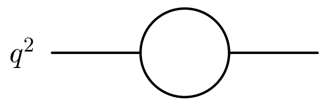


Figure 2: The one-loop master integral, $\mathrm{BB}(q^2)$.

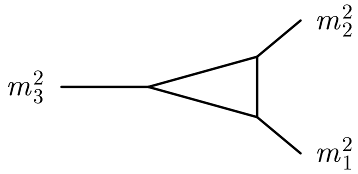


Figure 3: The one-loop master integral, $F_0(m_1^2, m_2^2, m_3^2)$.

These are not independent and can be reduced to the set of quadratic basis functions using the following relation obtained via integration by parts,

$$\begin{aligned}
&\int (x-x_0)^\alpha (x-x_1)^\beta G(v,\vec{w};x,y)dx = \\
&\quad = \frac{\alpha+\beta+2}{(\beta+1)(x_0-x_1)} \int (x-x_0)^\alpha (x-x_1)^{\beta+1} G(v,\vec{w};x,y)dx + \\
&\qquad + \frac{1}{(\beta+1)(x_0-x_1)} \int (x-x_0)^{\alpha+1} (x-x_1)^{\beta+1} f(v) G(\vec{w};x,y)dx - \\
&\qquad - \frac{(x-x_0)^{\alpha+1}(x-x_1)^{\beta+1}}{(\beta+1)(x_0-x_1)} G(v,\vec{w},;x,y)\,.
\end{aligned} \tag{4.29}$$

5. One loop

All one-loop master integrals contain an overall factor

$$S_D = \frac{(4\pi)^\epsilon}{16\pi^2} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{5.1}$$

5.1 Two point integrals

The only two point integral at one-loop level is the bubble graph shown in figure 2 which can be easily calculated using Feynman parameters and is given here for the sake of completeness,

$$\mathrm{BB}(q^2) = iS_D(-q^2)^{-\epsilon} \frac{1}{\epsilon} \frac{1}{(1-2\epsilon)}\,. \tag{5.2}$$

5.2 Three point integrals

The three point integrals with less than three massive legs are reducible to bubble integrals. The only master integral is the triangle with three massive legs, as shown in figure 3. The finite part of this diagram has been calculated in [51]. The d-dimensional result for this integral can be found in [8, 52, 53], where it is given in terms of Appell functions. Davydychev calculated this integral to all orders in terms of log-sine functions [47].

In two-loop calculations this integral appears in products with other one-loop integral. To be able to combine these integrals with the genuine two loop integrals, it is necessary to express all integrals in terms of the same set of functions. To achieve this we apply the differential equation method also to the one loop triangle.

The DE for $F_0(m_1^2, m_2^2, m_3^2)$ is given by

$$\begin{aligned} m_1^2 \Lambda^2 \frac{\partial F_0}{\partial m_1^2} = & \left(\left(\frac{d-4}{2} \right) \Lambda^2 + (3-d) m_1^2 (m_1^2 - m_2^2 - m_3^2) \right) F_0 + \\ & + (d-3)\,(m_3^2 + m_1^2 - m_2^2) \mathrm{BB}(m_3^2) + \\ & + (d-3)\,(m_1^2 + m_2^2 - m_3^2)\, \mathrm{BB}(m_2^2) - \\ & - 2\,(d-3) m_1^2\, \mathrm{BB}(m_1^2)\,, \end{aligned} \tag{5.3}$$

where,

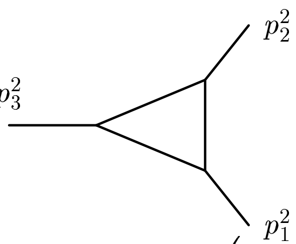
$$\Lambda^2 = m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2\,. \tag{5.4}$$

The scalar triangle is completely symmetric under the interchange of the external scales. We choose to solve it in the configuration $m_i^2 = p_i^2$ with $p_3^2 > p_1^2$, p_2^2. In this case, $\Lambda^2 = p_3^4 \lambda^2$ (with λ given by eq. (3.3)) and $m_1^2 = p_3^2 x$, $m_2^2 = p_3^2 y$. By performing a transformation of variable $m_1^2 \equiv x \to \lambda$ it can easily be seen that the homogeneous solution at $d = 4$ is $F_0^{hom} = \frac{1}{\lambda}$. To fix the constant of integration it is necessary to look for suitable boundary points. Note that the limit $x \to 0$ is not allowed because the one-loop triangle with two external masses is divergent at $d = 4$. The only remaining possibility is to choose a point on the parabola given by $\lambda = 0$, as shown in figure 1, i.e. $x \to x_{0,1} = (1 \pm \sqrt{y})^2$. In this case the homogeneous solution is divergent at the boundary and so we apply the treatment discussed in section 2. The boundary condition for $x \to x_0$, corresponding to $m_1^2 \to (m_2 - m_3)^2$ is given by,

$$F_0(m_1^2, m_2^2, m_3^2)|_{m_1^2=(m_2-m_3)^2} = -\frac{\mathrm{BB}({m_2}^2)}{(m_2 - m_3)\; m_3} + \frac{\mathrm{BB}((m_2 - m_3)^2)}{m_2\, m_3} + \frac{\mathrm{BB}({m_3}^2)}{m_2\; (m_2 - m_3)}. \tag{5.5}$$

Solving the differential equation order by order in ϵ, we find the following expansion in ϵ,

$F_0(p_1^2, p_2^2, p_3^2) =$



$$= i(-p_3^2)^{-1-\epsilon} S_D \frac{1}{\lambda} \left(\sum_{i=0,\dots,2} f_0^i \left(\frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2} \right) \epsilon^i + \mathcal{O}(\epsilon^3) \right), \tag{5.6}$$

where,

$$
\begin{aligned}
f_0^0(x,y) \;=\;& -2\,G(\lambda,0;x,y) - 2\,G(\lambda;x,y)\,H(0;x_0) + \\
& +G(\lambda;x,y)\,H(0;y) + (-1+y)\,G(x\lambda;x,y)\,H(0;y)\,, \\
f_0^1(x,y) \;=\;& +2\,G(0,\lambda,0;x,y) + 2\,G(\lambda,0,0;x,y) + 2\,G(x_0,\lambda,0;x,y) + \\
& +2\,G(x_1,\lambda,0;x,y) + 2\,G(\lambda;x,y)\,H(0,0;x_0) - (G(\lambda;x,y)\,H(0,0;y)) - \\
& +(1-y)\,G(x\lambda;x,y)\,H(0,0;y) + 2\,G(0,\lambda;x,y)\,H(0;x_0) - (G(0,\lambda;x,y)\,H(0;y)) + \\
& +(1-y)\,G(0,x\lambda;x,y)\,H(0;y) + 2\,G(\lambda,0;x,y)\,H(0;x_0) + 2\,G(x_0,\lambda;x,y)\,H(0;x_0) - \\
& -(G(x_0,\lambda;x,y)\,H(0;y)) + (1-y)\,G(x_0,x\lambda;x,y)\,H(0;y) + 2\,G(x_1,\lambda;x,y)\,H(0;x_0) - \\
& -(G(x_1,\lambda;x,y)\,H(0;y)) + (1-y)\,G(x_1,x\lambda;x,y)\,H(0;y) \\
f_0^2(x,y) \;=\;& -2\,G(0,0,\lambda,0;x,y) - 2\,G(0,\lambda,0,0;x,y) - 2\,G(0,x_0,\lambda,0;x,y) - \\
& -2\,G(0,x_1,\lambda,0;x,y) - 2\,G(\lambda,0,0,0;x,y) - 2\,G(x_0,0,\lambda,0;x,y) - \\
& -2\,G(x_0,\lambda,0,0;x,y) - 2\,G(x_0,x_0,\lambda,0;x,y) - 2\,G(x_0,x_1,\lambda,0;x,y) - \\
& -2\,G(x_1,0,\lambda,0;x,y) - 2\,G(x_1,\lambda,0,0;x,y) - 2\,G(x_1,x_0,\lambda,0;x,y) - \\
& -2\,G(x_1,x_1,\lambda,0;x,y) - 2\,G(\lambda;x,y)\,H(0,0,0;x_0) + G(\lambda;x,y)\,H(0,0,0;y) + \\
& +(-1+y)\,G(x\lambda;x,y)\,H(0,0,0;y) - 2\,G(0,\lambda;x,y)\,H(0,0;x_0) + \\
& +G(0,\lambda;x,y)\,H(0,0;y) + (-1+y)\,G(0,x\lambda;x,y)\,H(0,0;y) - \\
& -2\,G(\lambda,0;x,y)\,H(0,0;x_0) - 2\,G(x_0,\lambda;x,y)\,H(0,0;x_0) + \\
& +G(x_0,\lambda;x,y)\,H(0,0;y) + (-1+y)\,G(x_0,x\lambda;x,y)\,H(0,0;y) - \\
& -2\,G(x_1,\lambda;x,y)\,H(0,0;x_0) + G(x_1,\lambda;x,y)\,H(0,0;y) + \\
& +(-1+y)\,G(x_1,x\lambda;x,y)\,H(0,0;y) - 2\,G(0,0,\lambda;x,y)\,H(0;x_0) + \\
& +G(0,0,\lambda;x,y)\,H(0;y) + (-1+y)\,G(0,0,x\lambda;x,y)\,H(0;y) - \\
& -2\,G(0,\lambda,0;x,y)\,H(0;x_0) - 2\,G(0,x_0,\lambda;x,y)\,H(0;x_0) + \\
& +G(0,x_0,\lambda;x,y)\,H(0;y) + (-1+y)\,G(0,x_0,x\lambda;x,y)\,H(0;y) - \\
& -2\,G(0,x_1,\lambda;x,y)\,H(0;x_0) + G(0,x_1,\lambda;x,y)\,H(0;y) + \\
& +(-1+y)\,G(0,x_1,x\lambda;x,y)\,H(0;y) - 2\,G(\lambda,0,0;x,y)\,H(0;x_0) - \\
& -2\,G(x_0,0,\lambda;x,y)\,H(0;x_0) + G(x_0,0,\lambda;x,y)\,H(0;y) + \\
& +(-1+y)\,G(x_0,0,x\lambda;x,y)\,H(0;y) - 2\,G(x_0,\lambda,0;x,y)\,H(0;x_0) - \\
& -2\,G(x_0,x_0,\lambda;x,y)\,H(0;x_0) + G(x_0,x_0,\lambda;x,y)\,H(0;y) + \\
& +(-1+y)\,G(x_0,x_0,x\lambda;x,y)\,H(0;y) - 2\,G(x_0,x_1,\lambda;x,y)\,H(0;x_0) + \\
& +G(x_0,x_1,\lambda;x,y)\,H(0;y) + (-1+y)\,G(x_0,x_1,x\lambda;x,y)\,H(0;y) - \\
& -2\,G(x_1,0,\lambda;x,y)\,H(0;x_0) + G(x_1,0,\lambda;x,y)\,H(0;y) + \\
& +(-1+y)\,G(x_1,0,x\lambda;x,y)\,H(0;y) - 2\,G(x_1,\lambda,0;x,y)\,H(0;x_0) - \\
& -2\,G(x_1,x_0,\lambda;x,y)\,H(0;x_0) + G(x_1,x_0,\lambda;x,y)\,H(0;y) + \\
& +(-1+y)\,G(x_1,x_0,x\lambda;x,y)\,H(0;y) - 2\,G(x_1,x_1,\lambda;x,y)\,H(0;x_0) + \\
& +G(x_1,x_1,\lambda;x,y)\,H(0;y) + (-1+y)\,G(x_1,x_1,x\lambda;x,y)\,H(0;y)\,.
\end{aligned}
$$

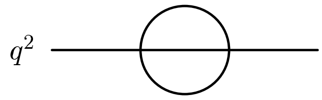


Figure 4: The two-loop master integral, $\mathrm{SS}(q^2)$.

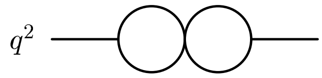


Figure 5: The two-loop master integral, $\mathrm{GL}(q^2)$.

We have checked that the ϵ^0 term agrees with the results of ref. [51] while the $\mathcal{O}(\epsilon)$ term numerically agrees with that given in ref. [47].

6. Two loop

6.1 Two point integrals

There are only two two-point integrals, both of which can be obtained by repeated one-loop integration and are widely available in the literature. We quote them here for the sake of completeness.

The three-propagator sunset graph $\mathrm{SS}(q^2)$ is shown in figure 4 and is given by,

$$\begin{aligned}\mathrm{SS}(q^2) &= S_D^2(-q^2)^{1-2\epsilon}\frac{\Gamma(2\epsilon-1)\Gamma(1-2\epsilon)^2}{\Gamma(3-3\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon)^2} \\ &= S_D^2(-q^2)^{1-2\epsilon}\left[-\frac{1}{4\epsilon}-\frac{13}{8}-\frac{115}{16}\epsilon-\left(\frac{865}{32}-\frac{3}{2}\zeta(3)\right)\epsilon^2+\mathcal{O}(\epsilon^3)\right]. \end{aligned} \tag{6.1}$$

We denote the four-propagator graph shown in Fig. 5 by $\mathrm{GL}(q^2)$. It is given by,

$$\begin{aligned}\mathrm{GL}(q^2) = \mathrm{BB}(q^2)^2 &= -(-q^2)^{-2\epsilon}S_D^2\frac{1}{\epsilon^2(1-2\epsilon)^2} \\ &= -(-q^2)^{-2\epsilon}S_D^2\frac{1}{\epsilon^2}\left(1+4\epsilon+12\epsilon^2+32\epsilon^3+80\epsilon^4+\mathcal{O}(\epsilon^5)\right). \end{aligned} \tag{6.2}$$

6.2 Three point integrals

6.2.1 Master integrals with four propagators

The simplest graph with four propagators is denoted by TGL and is shown in figure 6 and is simply the product of two bubbles,

$$\mathrm{TGL}(p_1^2,p_2^2) = \mathrm{BB}(p_1^2)\mathrm{BB}(p_2^2) = -(-p_1^2)^{-\epsilon}(-p_2^2)^{-\epsilon}S_D^2\frac{1}{\epsilon^2(1-2\epsilon)^2}\,. \tag{6.3}$$

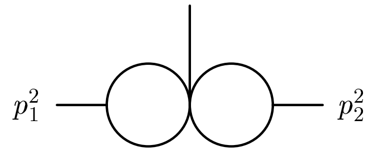


Figure 6: The two-loop master integral, $\mathrm{TGL}(p_1^2, p_2^2)$.

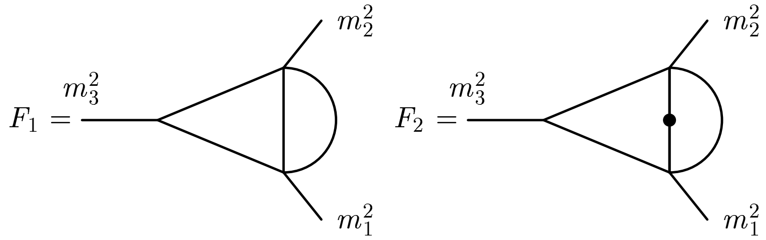


Figure 7: The two-loop master integrals, $F_1(m_1^2, m_2^2, m_3^2)$, $F_2(m_1^2, m_2^2, m_3^2)$.

There are two genuine two loop integrals with four propagators. These are denoted by F_1 and F_2 and shown in figure 7. The ϵ expansion of these integrals is given to order ϵ^0 in [54]. The corresponding two scale diagrams, when one of the legs becomes massless, are given in [15] to order ϵ^2 in terms of HPL's.

We obtain two coupled differential equations for F_1 and F_2,

$$\begin{aligned}
\frac{\partial}{\partial m_1^2} F_1(m_1^2, m_2^2, m_3^2) &= \frac{(d-4)\ m_2^2\ \left(m_1^2 - m_2^2 + m_3^2\right)}{2\ (d-3)\ \Lambda^2} F_2 \ - \\
&\quad \frac{(d-4)\ \left(-m_1^2 + m_2^2 + m_3^2\right)}{2\,\Lambda^2} F_1 \ + \\
&\quad + \frac{(3\,d-8)\ \left(-m_1^2 - m_2^2 + m_3^2\right)}{2\,\Lambda^2\, m_1^2} \mathrm{SS}(m_1^2) + \frac{(3\,d-8)}{\Lambda^2}\,\mathrm{SS}(m_2^2)\,, \qquad (6.4)\\
\frac{\partial}{\partial m_1^2} F_2(m_1^2, m_2^2, m_3^2) &= \frac{-\left((d-3)\ (3\,d-10)\ \left(m_1^2 - m_2^2 + m_3^2\right)\right)}{2\,\Lambda^2\, m_1^2} F_1 \ - \\
&\quad - \frac{1}{2\,\Lambda^2\, m_1^2} \Big((3d-10) m_1^2 (m_1^2 - m_2^2 - m_3^2) - 2(d-4)\Lambda^2 \Big) F_2 \ + \\
&\quad + \frac{(d-3)\ (3\,d-8)\ (3\,d-10)}{(d-4)\ \Lambda^2\, m_1^2}\,\mathrm{SS}(m_1^2) \ - \\
&\quad - \frac{(d-3)\ (3\,d-8)\ (3\,d-10)\ \left(m_1^2 + m_2^2 - m_3^2\right)}{2\ (d-4)\ \Lambda^2\, m_1^2\, m_2^2}\,\mathrm{SS}(m_2^2)\,. \qquad (6.5)
\end{aligned}$$

The homogeneous solutions at $d = 4$ are found to be

$$F_1^{hom} = 1, \qquad F_2^{hom} = \frac{1}{\lambda}.$$

Finding suitable boundary conditions for these coupled equations is more difficult than for the one loop triangle. Only F_1 has a smooth limit for $x \to 0$ (F_2 develops additional poles in ϵ in that limit). On the other hand, taking the limit $\lambda \to 0$ in either differential equation only provides a single relation between the two integrals. Therefore we use the limit $x \to 0$ to fix the integration constant of F_1, matching the integral in this limit to the two-scale result given in [15]. Then we take the limit $\lambda \to 0$ (i.e. $x \to x_0$) of F_1 and use the limiting relation obtained from the differential equation to find F_2 in the limit $\lambda \to 0$, thereby fixing the integration constant of F_2.

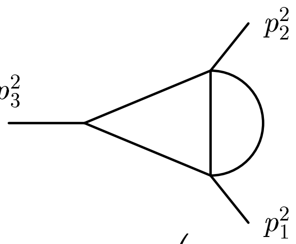
The limiting relation in the limit $x \to x_0$ is,

$$
\begin{aligned}
F_2(m_1^2, m_2^2, m_3^2)|_{m_1^2=(m_2-m_3)^2} = & -\left(\frac{(d-3)\ F_1(m_1^2, m_2^2, m_3^2)|_{m_1^2=(m_2-m_3)^2}}{m_2\, m_3}\right) + \\
& + \frac{(d-3)\,(3\,d-8)\,\mathrm{SS}({m_2}^2)}{(d-4)\ {m_2}^2\,(m_2-m_3)\,m_3} - \frac{(d-3)\,(3\,d-8)\,\mathrm{SS}(m_3^2)}{(d-4)\ m_2\,m_3^2\,(m_2-m_3)}.
\end{aligned}
\tag{6.6}
$$

Without the second boundary constraint on F_1 as $x \to 0$, this would only be sufficient to yield an independent determination of f_1^{-2} only.

The expansions in ϵ are needed for two separate configurations. First, when the largest scale p_3^2 is situated opposite the bubble (corresponding to $m_1^2 = p_3^2$) and second, when it lies adjacent to the bubble ($m_2^2 = p_3^2$). For the first momentum configuration, we find that the expansions are as follows,

$F_1^a(p_1^2, p_2^2, p_3^2) =$



$$
= S_D^2(-p_3^2)^{-2\epsilon}\left(\sum_{i=-2,\ldots,1} f_1^i\left(\frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2}\right)\epsilon^i + \mathcal{O}(\epsilon^2)\right),
$$

where,

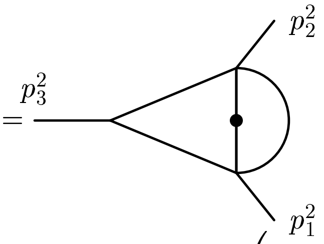
$$
\begin{aligned}
f_1^{-2}(x,y) &= -\frac{1}{2}, \\
f_1^{-1}(x,y) &= -\frac{5}{2}, \\
f_1^{0}(x,y) &= \frac{-57-\pi^2}{6} - H(1,0;y) + \frac{-1+x+y}{\lambda}\,G(\lambda,0;x,y) + G(\lambda,0;0,y) - \\
& - \frac{1}{2}\,\Delta G(0;0,y)\,H(0;y) + \frac{1}{2}\,G(0;x,y)\,H(0;y) + \\
& + \frac{-\left((-1+y)\ (-1+x+y)\right)}{2\,\lambda}\,G(x\lambda;x,y)\,H(0;y) + \\
& + G(\lambda;0,y)\,H(0;x_0) - \frac{1}{2}\,G(\lambda;0,y)\,H(0;y) - \\
& + \frac{-(-1+x+y)}{2\,\lambda}\,G(\lambda;x,y)\,H(0;y) + \frac{-1+x+y}{\lambda}\,G(\lambda;x,y)\,H(0;x_0), \\
f_1^{1}(x,y) &= -\left(\frac{65}{2}\right) - \frac{5\,\pi^2}{6} + 2\,\zeta_3 + \frac{-\pi^2}{12}\,\Delta G(0;0,y) - \Delta G(0,\lambda,0;0,y) + \\
& + \frac{\pi^2}{12}\,G(0;x,y) + \frac{\pi^2}{12}\,G(\lambda;0,y) + \frac{\pi^2\ (-1+x+y)}{12\,\lambda}\,G(\lambda;x,y) +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{-\left(\pi^2\ (-1+y)\ (-1+x+y)\right)}{12\,\lambda}\,G(x\lambda;x,y)+5\,G(\lambda,0;0,y)+\\
&+\frac{5\ (-1+x+y)}{\lambda}\,G(\lambda,0;x,y)+\frac{-3\ (-1+x+y)}{2\,\lambda}\,G(0,\lambda,0;x,y)-\\
&-G(\lambda,0,0;0,y)+\frac{-2\ (-1+x+y)}{\lambda}\,G(\lambda,0,0;x,y)-\\
&-\frac{1}{2}\,G(\lambda,0,\lambda;0,y)+\frac{-1+y}{2}\,G(\lambda,0,x\lambda;0,y)+\frac{1}{2}\,G(\lambda,\lambda,0;0,y)-\\
&-\frac{3}{2}\,G(\lambda,\lambda,0;x,y)+\frac{-1+y}{2}\,G(\lambda,x\lambda,0;0,y)-G(x_0,\lambda,0;0,y)-\\
&-\left(\frac{-1+x+y}{\lambda}\right)\,G(x_0,\lambda,0;x,y)-G(x_1,\lambda,0;0,y)-\\
&-\left(\frac{-1+x+y}{\lambda}\right)\,G(x_1,\lambda,0;x,y)+\frac{3\ (-1+y)}{2}\,G(x\lambda,\lambda,0;x,y)+\\
&+\frac{-\pi^2}{6}\,H(1;y)-5\,H(1,0;y)+2\,H(1,0,0;y)-H(1,1,0;y)-\\
&-\frac{5}{2}\,\Delta G(0;0,y)\,H(0;y)+\Delta G(0;0,y)\,H(0,0;y)-\\
&-\frac{1}{2}\,\Delta G(0;0,y)\,H(1,0;y)-\frac{1}{4}\,\Delta G(0,0;0,y)\,H(0;y)-\\
&-\Delta G(0,\lambda;0,y)\,H(0;x_0)+\frac{3}{4}\,\Delta G(0,\lambda;0,y)\,H(0;y)+\\
&+\frac{-1+y}{4}\,\Delta G(0,x\lambda;0,y)\,H(0;y)-\frac{1}{2}\,G(0;x,y)\,G(\lambda,0;0,y)+\\
&+\frac{5}{2}\,G(0;x,y)\,H(0;y)--G(0;x,y)\,H(0,0;y)++\frac{1}{2}\,G(0;x,y)\,H(1,0;y)+\\
&+5\,G(\lambda;0,y)\,H(0;x_0)-\frac{5}{2}\,G(\lambda;0,y)\,H(0;y)-2\,G(\lambda;0,y)\,H(0,0;x_0)+\\
&+G(\lambda;0,y)\,H(0,0;y)+\frac{1}{2}\,G(\lambda;0,y)\,H(1,0;y)+\\
&+\frac{-(-1+x+y)}{2\,\lambda}\,G(\lambda;x,y)\,G(\lambda,0;0,y)+\frac{5\ (-1+x+y)}{\lambda}\,G(\lambda;x,y)\,H(0;x_0)+\\
&+\frac{-5\ (-1+x+y)}{2\,\lambda}\,G(\lambda;x,y)\,H(0;y)+\frac{-2\ (-1+x+y)}{\lambda}\,G(\lambda;x,y)\,H(0,0;x_0)+\\
&+\frac{-1+x+y}{\lambda}\,G(\lambda;x,y)\,H(0,0;y)+\frac{-1+x+y}{2\,\lambda}\,G(\lambda;x,y)\,H(1,0;y)+\\
&+\frac{(-1+y)\ (-1+x+y)}{2\,\lambda}\,G(x\lambda;x,y)\,G(\lambda,0;0,y)+\\
&+\frac{-5\ (-1+y)\ (-1+x+y)}{2\,\lambda}\,G(x\lambda;x,y)\,H(0;y)+\\
&+\frac{(-1+y)\ (-1+x+y)}{\lambda}\,G(x\lambda;x,y)\,H(0,0;y)+\\
&+\frac{-((-1+y)\ (-1+x+y))}{2\,\lambda}\,G(x\lambda;x,y)\,H(1,0;y)-\\
&-\frac{1}{4}\,G(0,0;x,y)\,H(0;y)+\frac{-3\ (-1+x+y)}{2\,\lambda}\,G(0,\lambda;x,y)\,H(0;x_0)+\\
&+\frac{3\ (-1+x+y)}{4\,\lambda}\,G(0,\lambda;x,y)\,H(0;y)+\\
&+\frac{3\ (-1+y)\ (-1+x+y)}{4\,\lambda}\,G(0,x\lambda;x,y)\,H(0;y)-\\
&-\frac{3}{2}\,G(\lambda,0;0,y)\,H(0;x_0)-\frac{1}{4}\,G(\lambda,0;0,y)\,H(0;y)+\\
&+\frac{-2\ (-1+x+y)}{\lambda}\,G(\lambda,0;x,y)\,H(0;x_0)+
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{-(-1+x+y)}{4\,\lambda}\,G(\lambda,0;x,y)\,H(0;y) + \\
&+ \frac{1}{2}\,G(\lambda,\lambda;0,y)\,H(0;x_0) - \frac{1}{4}\,G(\lambda,\lambda;0,y)\,H(0;y) - \\
&- \frac{3}{2}\,G(\lambda,\lambda;x,y)\,H(0;x_0) + \frac{3}{4}\,G(\lambda,\lambda;x,y)\,H(0;y) + \\
&+ \frac{-1+y}{2}\,G(\lambda,x\lambda;0,y)\,H(0;x_0) + \frac{-3\,(-1+y)}{4}\,G(\lambda,x\lambda;0,y)\,H(0;y) + \\
&+ \frac{3\,(-1+y)}{4}\,G(\lambda,x\lambda;x,y)\,H(0;y) - (G(x_0,\lambda;0,y)\,H(0;x_0)) + \\
&+ \frac{1}{2}\,G(x_0,\lambda;0,y)\,H(0;y) - \left(\frac{-1+x+y}{\lambda}\right)\,G(x_0,\lambda;x,y)\,H(0;x_0) + \\
&+ \frac{-1+x+y}{2\,\lambda}\,G(x_0,\lambda;x,y)\,H(0;y) + \frac{-1+y}{2}\,G(x_0,x\lambda;0,y)\,H(0;y) + \\
&+ \frac{(-1+y)\,(-1+x+y)}{2\,\lambda}\,G(x_0,x\lambda;x,y)\,H(0;y) - (G(x_1,\lambda;0,y)\,H(0;x_0)) + \\
&+ \frac{1}{2}\,G(x_1,\lambda;0,y)\,H(0;y) - \left(\frac{-1+x+y}{\lambda}\right)\,G(x_1,\lambda;x,y)\,H(0;x_0) + \\
&+ \frac{-1+x+y}{2\,\lambda}\,G(x_1,\lambda;x,y)\,H(0;y) + \frac{-1+y}{2}\,G(x_1,x\lambda;0,y)\,H(0;y) + \\
&+ \frac{(-1+y)\,(-1+x+y)}{2\,\lambda}\,G(x_1,x\lambda;x,y)\,H(0;y) + \\
&+ \frac{(-1+y)\,(-1+x+y)}{4\,\lambda}\,G(x\lambda,0;x,y)\,H(0;y) + \\
&+ \frac{3\,(-1+y)}{2}\,G(x\lambda,\lambda;x,y)\,H(0;x_0) + \\
&+ \frac{-3\,(-1+y)}{4}\,G(x\lambda,\lambda;x,y)\,H(0;y) + \frac{-3\,(-1+y)^2}{4}\,G(x\lambda,x\lambda;x,y)\,H(0;y) + \\
&+ \frac{1}{4}\,\Delta G(0;0,y)\,G(0;x,y)\,H(0;y) + \frac{-1+x+y}{4\,\lambda}\,\Delta G(0;0,y)\,G(\lambda;x,y)\,H(0;y) - \\
&+ \frac{-((-1+y)\,(-1+x+y))}{4\,\lambda}\,\Delta G(0;0,y)\,G(x\lambda;x,y)\,H(0;y) - \\
&- \frac{1}{2}\,G(0;x,y)\,G(\lambda;0,y)\,H(0;x_0) + \frac{1}{4}\,G(0;x,y)\,G(\lambda;0,y)\,H(0;y) + \\
&+ \frac{-(-1+x+y)}{2\,\lambda}\,G(\lambda;0,y)\,G(\lambda;x,y)\,H(0;x_0) + \\
&+ \frac{-1+x+y}{4\,\lambda}\,G(\lambda;0,y)\,G(\lambda;x,y)\,H(0;y) + \\
&+ \frac{(-1+y)\,(-1+x+y)}{2\,\lambda}\,G(\lambda;0,y)\,G(x\lambda;x,y)\,H(0;x_0) + \\
&+ \frac{-((-1+y)\,(-1+x+y))}{4\,\lambda}\,G(\lambda;0,y)\,G(x\lambda;x,y)\,H(0;y)\,,
\end{aligned}
$$

and,

$F_2^a(p_1^2,p_2^2,p_3^2) =$



$$
= S_D^2(-p_3^2)^{-1-2\epsilon}\frac{1}{\lambda}\left(\sum_{i=-1,\dots,1} f_2^i\left(\frac{p_1^2}{p_3^2},\frac{p_2^2}{p_3^2}\right)\epsilon^i + \mathcal{O}(\epsilon^2)\right),
$$

where,

$$
\begin{aligned}
f_2^{-1}(x,y) \;=\; & -2\,G(\lambda,0;x,y) - 2\,G(\lambda;x,y)\,H(0;x_0) + \\
& + G(\lambda;x,y)\,H(0;y) + (-1+y)\,G(x\lambda;x,y)\,H(0;y)\,, \\
f_2^{0}(x,y) \;=\; & + \frac{-\pi^2}{6}\,G(\lambda;x,y) + \frac{\pi^2\,(-1+y)}{6}\,G(x\lambda;x,y) + 3\,G(0,\lambda,0;x,y) + \\
& +4\,G(\lambda,0,0;x,y) + 2\,G(x_0,\lambda,0;x,y) + 2\,G(x_1,\lambda,0;x,y) + \\
& + G(\lambda;x,y)\,G(\lambda,0;0,y) + 4G(\lambda;x,y)\,H(0,0;x_0) - 2G(\lambda;x,y)\,H(0,0;y) - \\
& - (G(\lambda;x,y)\,H(1,0;y)) + (1-y)\,G(x\lambda;x,y)\,G(\lambda,0;0,y) + \\
& + (2-2\,y)\,G(x\lambda;x,y)\,H(0,0;y) + (-1+y)\,G(x\lambda;x,y)\,H(1,0;y) + \\
& + 3\,G(0,\lambda;x,y)\,H(0;x_0) - \frac{3}{2}\,G(0,\lambda;x,y)\,H(0;y) + \\
& + \frac{-3\,(-1+y)}{2}\,G(0,x\lambda;x,y)\,H(0;y) + 4\,G(\lambda,0;x,y)\,H(0;x_0) + \\
& + \frac{1}{2}\,G(\lambda,0;x,y)\,H(0;y) + 2\,G(x_0,\lambda;x,y)\,H(0;x_0) - \\
& - \,(G(x_0,\lambda;x,y)\,H(0;y)) + \\
& + (1-y)\,G(x_0,x\lambda;x,y)\,H(0;y) + 2\,G(x_1,\lambda;x,y)\,H(0;x_0) - \\
& - \,(G(x_1,\lambda;x,y)\,H(0;y)) + \\
& + (1-y)\,G(x_1,x\lambda;x,y)\,H(0;y) + \frac{1-y}{2}\,G(x\lambda,0;x,y)\,H(0;y) - \\
& - \frac{1}{2}\,\Delta G(0;0,y)\,G(\lambda;x,y)\,H(0;y) + \\
& + \frac{-1+y}{2}\,\Delta G(0;0,y)\,G(x\lambda;x,y)\,H(0;y) + \\
& + G(\lambda;0,y)\,G(\lambda;x,y)\,H(0;x_0) - \frac{1}{2}\,G(\lambda;0,y)\,G(\lambda;x,y)\,H(0;y) + \\
& + (1-y)\,G(\lambda;0,y)\,G(x\lambda;x,y)\,H(0;x_0) + \\
& + \frac{-1+y}{2}\,G(\lambda;0,y)\,G(x\lambda;x,y)\,H(0;y)\,, \\
f_2^{1}(x,y) \;=\; & -\zeta_3\,G(\lambda;x,y) + (-1+y)\;\zeta_3\,G(x\lambda;x,y) + \frac{\pi^2}{4}\,G(0,\lambda;x,y) + \\
& + \frac{-\left(\pi^2\,(-1+y)\right)}{4}\,G(0,x\lambda;x,y) + \frac{\pi^2}{12}\,G(\lambda,0;x,y) + \frac{\pi^2}{6}\,G(x_0,\lambda;x,y) + \\
& + \frac{-\left(\pi^2\,(-1+y)\right)}{6}\,G(x_0,x\lambda;x,y) + \frac{\pi^2}{6}\,G(x_1,\lambda;x,y) + \\
& + \frac{-\left(\pi^2\,(-1+y)\right)}{6}\,G(x_1,x\lambda;x,y) + \frac{-\left(\pi^2\,(-1+y)\right)}{12}\,G(x\lambda,0;x,y) - \\
& - \frac{9}{2}\,G(0,0,\lambda,0;x,y) - 6\,G(0,\lambda,0,0;x,y) - 3\,G(0,x_0,\lambda,0;x,y) - \\
& - 3\,G(0,x_1,\lambda,0;x,y) - 8\,G(\lambda,0,0,0;x,y) - \frac{3}{2}\,G(\lambda,\lambda,\lambda,0;x,y) + \\
& + \frac{3\,(-1+y)}{2}\,G(\lambda,x\lambda,\lambda,0;x,y) - 3\,G(x_0,0,\lambda,0;x,y) - \\
& - 4\,G(x_0,\lambda,0,0;x,y) - 2\,G(x_0,x_0,\lambda,0;x,y) -
\end{aligned}
$$

$$
\begin{aligned}
&-2\,G(x_0,x_1,\lambda,0;x,y)-3\,G(x_1,0,\lambda,0;x,y)-\\
&-4\,G(x_1,\lambda,0,0;x,y)-2\,G(x_1,x_0,\lambda,0;x,y)-2\,G(x_1,x_1,\lambda,0;x,y)+\\
&+\frac{3\,(-1+y)}{2}\,G(x\lambda,\lambda,\lambda,0;x,y)+\frac{-3\,(-1+y)^2}{2}\,G(x\lambda,x\lambda,\lambda,0;x,y)+\\
&+\frac{-\pi^2}{12}\,\Delta G(0;0,y)\,G(\lambda;x,y)+\frac{\pi^2\,(-1+y)}{12}\,\Delta G(0;0,y)\,G(x\lambda;x,y)-\\
&-\left(\Delta G(0,\lambda,0;0,y)\,G(\lambda;x,y)\right)+(-1+y)\,\Delta G(0,\lambda,0;0,y)\,G(x\lambda;x,y)+\\
&+\frac{\pi^2}{12}\,G(\lambda;0,y)\,G(\lambda;x,y)+\frac{-\left(\pi^2\,(-1+y)\right)}{12}\,G(\lambda;0,y)\,G(x\lambda;x,y)-\\
&-\left(G(\lambda;x,y)\,G(\lambda,0,0;0,y)\right)-\frac{1}{2}\,G(\lambda;x,y)\,G(\lambda,0,\lambda;0,y)+\\
&+\frac{-1+y}{2}\,G(\lambda;x,y)\,G(\lambda,0,x\lambda;0,y)+\frac{1}{2}\,G(\lambda;x,y)\,G(\lambda,\lambda,0;0,y)+\\
&+\frac{-1+y}{2}\,G(\lambda;x,y)\,G(\lambda,x\lambda,0;0,y)-\left(G(\lambda;x,y)\,G(x_0,\lambda,0;0,y)\right)-\\
&-\left(G(\lambda;x,y)\,G(x_1,\lambda,0;0,y)\right)+\frac{-\pi^2}{6}\,G(\lambda;x,y)\,H(1;y)-\\
&-8\,G(\lambda;x,y)\,H(0,0,0;x_0)+4\,G(\lambda;x,y)\,H(0,0,0;y)+\\
&+2\,G(\lambda;x,y)\,H(1,0,0;y)-\left(G(\lambda;x,y)\,H(1,1,0;y)\right)+\\
&+(-1+y)G(x\lambda;x,y)G(\lambda,0,0;0,y)+\frac{-1+y}{2}G(x\lambda;x,y)G(\lambda,0,\lambda;0,y)+\\
&+\frac{-(-1+y)^2}{2}\,G(x\lambda;x,y)\,G(\lambda,0,x\lambda;0,y)+\\
&+\frac{1-y}{2}\,G(x\lambda;x,y)\,G(\lambda,\lambda,0;0,y)+\\
&+\frac{-(-1+y)^2}{2}\,G(x\lambda;x,y)\,G(\lambda,x\lambda,0;0,y)+\\
&+(-1+y)\,G(x\lambda;x,y)\,G(x_0,\lambda,0;0,y)+\\
&+(-1+y)\,G(x\lambda;x,y)\,G(x_1,\lambda,0;0,y)+\frac{\pi^2\,(-1+y)}{6}\,G(x\lambda;x,y)\,H(1;y)+\\
&+4\,(-1+y)\,G(x\lambda;x,y)\,H(0,0,0;y)+(2-2\,y)\,G(x\lambda;x,y)\,H(1,0,0;y)+\\
&+(-1+y)\,G(x\lambda;x,y)\,H(1,1,0;y)-\frac{3}{2}\,G(0,\lambda;x,y)\,G(\lambda,0;0,y)-\\
&-6\,G(0,\lambda;x,y)\,H(0,0;x_0)+3\,G(0,\lambda;x,y)\,H(0,0;y)+\\
&+\frac{3}{2}\,G(0,\lambda;x,y)\,H(1,0;y)+\frac{3\,(-1+y)}{2}\,G(0,x\lambda;x,y)\,G(\lambda,0;0,y)+\\
&+3\,(-1+y)\,G(0,x\lambda;x,y)\,H(0,0;y)+\\
&+\frac{-3\,(-1+y)}{2}\,G(0,x\lambda;x,y)\,H(1,0;y)-\\
&-\frac{1}{2}\,G(\lambda,0;0,y)\,G(\lambda,0;x,y)-\left(G(\lambda,0;0,y)\,G(x_0,\lambda;x,y)\right)+\\
&+(-1+y)\,G(\lambda,0;0,y)\,G(x_0,x\lambda;x,y)-\left(G(\lambda,0;0,y)\,G(x_1,\lambda;x,y)\right)+\\
&+(-1+y)\,G(\lambda,0;0,y)\,G(x_1,x\lambda;x,y)+\\
&+\frac{-1+y}{2}\,G(\lambda,0;0,y)\,G(x\lambda,0;x,y)-
\end{aligned}
$$

$$
\begin{aligned}
&- 8\,G(\lambda,0;x,y)\,H(0,0;x_0) - (G(\lambda,0;x,y)\,H(0,0;y)) + \\
&+ \frac{1}{2}\,G(\lambda,0;x,y)\,H(1,0;y) - 4\,G(x_0,\lambda;x,y)\,H(0,0;x_0) + \\
&+ 2\,G(x_0,\lambda;x,y)\,H(0,0;y) + G(x_0,\lambda;x,y)\,H(1,0;y) + \\
&+ 2\,(-1+y)\;G(x_0,x\lambda;x,y)\,H(0,0;y) + \\
&+ (1-y)\,G(x_0,x\lambda;x,y)\,H(1,0;y) - \\
&- 4\,G(x_1,\lambda;x,y)\,H(0,0;x_0) + 2\,G(x_1,\lambda;x,y)\,H(0,0;y) + \\
&+ G(x_1,\lambda;x,y)\,H(1,0;y) + \\
&+ 2\,(-1+y)\;G(x_1,x\lambda;x,y)\,H(0,0;y) + \\
&+ (1-y)\,G(x_1,x\lambda;x,y)\,H(1,0;y) + \\
&+ (-1+y)\,G(x\lambda,0;x,y)\,H(0,0;y) + \frac{1-y}{2}\,G(x\lambda,0;x,y)\,H(1,0;y) - \\
&- \frac{9}{2}\,G(0,0,\lambda;x,y)\,H(0;x_0) + \frac{9}{4}\,G(0,0,\lambda;x,y)\,H(0;y) + \\
&+ \frac{9\,(-1+y)}{4}\,G(0,0,x\lambda;x,y)\,H(0;y) - 6\,G(0,\lambda,0;x,y)\,H(0;x_0) - \\
&- \frac{3}{4}\,G(0,\lambda,0;x,y)\,H(0;y) - 3\,G(0,x_0,\lambda;x,y)\,H(0;x_0) + \\
&+ \frac{3}{2}\,G(0,x_0,\lambda;x,y)\,H(0;y) + \frac{3\,(-1+y)}{2}\,G(0,x_0,x\lambda;x,y)\,H(0;y) - \\
&- 3\,G(0,x_1,\lambda;x,y)\,H(0;x_0) + \frac{3}{2}\,G(0,x_1,\lambda;x,y)\,H(0;y) + \\
&+ \frac{3\,(-1+y)}{2}\,G(0,x_1,x\lambda;x,y)\,H(0;y) + \\
&+ \frac{3\,(-1+y)}{4}\,G(0,x\lambda,0;x,y)\,H(0;y) - \\
&- 8\,G(\lambda,0,0;x,y)\,H(0;x_0) - \frac{1}{4}\,G(\lambda,0,0;x,y)\,H(0;y) - \\
&- \frac{3}{2}\,G(\lambda,\lambda,\lambda;x,y)\,H(0;x_0) + \frac{3}{4}\,G(\lambda,\lambda,\lambda;x,y)\,H(0;y) + \\
&+ \frac{3\,(-1+y)}{4}\,G(\lambda,\lambda,x\lambda;x,y)\,H(0;y) + \\
&+ \frac{3\,(-1+y)}{2}\,G(\lambda,x\lambda,\lambda;x,y)\,H(0;x_0) + \\
&+ \frac{-3\,(-1+y)}{4}\,G(\lambda,x\lambda,\lambda;x,y)\,H(0;y) + \\
&+ \frac{-3\,(-1+y)^2}{4}\,G(\lambda,x\lambda,x\lambda;x,y)\,H(0;y) - \\
&- 3\,G(x_0,0,\lambda;x,y)\,H(0;x_0) + \frac{3}{2}\,G(x_0,0,\lambda;x,y)\,H(0;y) + \\
&+ \frac{3\,(-1+y)}{2}\,G(x_0,0,x\lambda;x,y)\,H(0;y) - 4\,G(x_0,\lambda,0;x,y)\,H(0;x_0) - \\
&- \frac{1}{2}\,G(x_0,\lambda,0;x,y)\,H(0;y) - 2\,G(x_0,x_0,\lambda;x,y)\,H(0;x_0) + \\
&+ G(x_0,x_0,\lambda;x,y)\,H(0;y) + (-1+y)\,G(x_0,x_0,x\lambda;x,y)\,H(0;y) -
\end{aligned}
$$

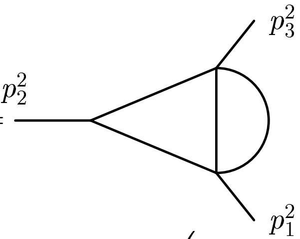
$$
\begin{aligned}
&-2\,G(x_0,x_1,\lambda;x,y)\,H(0;x_0)+G(x_0,x_1,\lambda;x,y)\,H(0;y)+\\
&+(-1+y)\,G(x_0,x_1,x\lambda;x,y)\,H(0;y)+\\
&+\frac{-1+y}{2}\,G(x_0,x\lambda,0;x,y)\,H(0;y)-\\
&-3\,G(x_1,0,\lambda;x,y)\,H(0;x_0)+\frac{3}{2}\,G(x_1,0,\lambda;x,y)\,H(0;y)+\\
&+\frac{3\;(-1+y)}{2}\,G(x_1,0,x\lambda;x,y)\,H(0;y)-4\,G(x_1,\lambda,0;x,y)\,H(0;x_0)-\\
&-\frac{1}{2}\,G(x_1,\lambda,0;x,y)\,H(0;y)-2\,G(x_1,x_0,\lambda;x,y)\,H(0;x_0)+\\
&+G(x_1,x_0,\lambda;x,y)\,H(0;y)+(-1+y)\,G(x_1,x_0,x\lambda;x,y)\,H(0;y)-\\
&-2\,G(x_1,x_1,\lambda;x,y)\,H(0;x_0)+\\
&+G(x_1,x_1,\lambda;x,y)\,H(0;y)+(-1+y)\,G(x_1,x_1,x\lambda;x,y)\,H(0;y)+\\
&+\frac{-1+y}{2}\,G(x_1,x\lambda,0;x,y)\,H(0;y)+\frac{-1+y}{4}\,G(x\lambda,0,0;x,y)\,H(0;y)+\\
&+\frac{3\;(-1+y)}{2}\,G(x\lambda,\lambda,\lambda;x,y)\,H(0;x_0)+\\
&+\frac{-3\;(-1+y)}{4}\,G(x\lambda,\lambda,\lambda;x,y)\,H(0;y)+\\
&+\frac{-3\,(-1+y)^2}{4}\,G(x\lambda,\lambda,x\lambda;x,y)\,H(0;y)+\\
&+\frac{-3\,(-1+y)^2}{2}\,G(x\lambda,x\lambda,\lambda;x,y)\,H(0;x_0)+\\
&+\frac{3\,(-1+y)^2}{4}\,G(x\lambda,x\lambda,\lambda;x,y)\,H(0;y)+\\
&+\frac{3\,(-1+y)^3}{4}\,G(x\lambda,x\lambda,x\lambda;x,y)\,H(0;y)+\\
&+\Delta G(0;0,y)\,G(\lambda;x,y)\,H(0,0;y)-\frac{1}{2}\,\Delta G(0;0,y)\,G(\lambda;x,y)\,H(1,0;y)+\\
&+(1-y)\,\Delta G(0;0,y)\,G(x\lambda;x,y)\,H(0,0;y)+\\
&+\frac{-1+y}{2}\,\Delta G(0;0,y)\,G(x\lambda;x,y)\,H(1,0;y)+\\
&+\frac{3}{4}\,\Delta G(0;0,y)\,G(0,\lambda;x,y)\,H(0;y)+\\
&+\frac{-3\;(-1+y)}{4}\,\Delta G(0;0,y)\,G(0,x\lambda;x,y)\,H(0;y)+\\
&+\frac{1}{4}\,\Delta G(0;0,y)\,G(\lambda,0;x,y)\,H(0;y)+\frac{1}{2}\,\Delta G(0;0,y)\,G(x_0,\lambda;x,y)\,H(0;y)+\\
&+\frac{1-y}{2}\,\Delta G(0;0,y)\,G(x_0,x\lambda;x,y)\,H(0;y)+\\
&+\frac{1}{2}\,\Delta G(0;0,y)\,G(x_1,\lambda;x,y)\,H(0;y)+\\
&+\frac{1-y}{2}\,\Delta G(0;0,y)\,G(x_1,x\lambda;x,y)\,H(0;y)+\\
&+\frac{1-y}{4}\,\Delta G(0;0,y)\,G(x\lambda,0;x,y)\,H(0;y)-
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{4}\,\Delta G(0,0;0,y)\,G(\lambda;x,y)\,H(0;y)+\\
&+\frac{-1+y}{4}\,\Delta G(0,0;0,y)\,G(x\lambda;x,y)\,H(0;y)-\\
&-\left(\Delta G(0,\lambda;0,y)\,G(\lambda;x,y)\,H(0;x_0)\right)+\frac{3}{4}\,\Delta G(0,\lambda;0,y)\,G(\lambda;x,y)\,H(0;y)+\\
&+(-1+y)\,\Delta G(0,\lambda;0,y)\,G(x\lambda;x,y)\,H(0;x_0)+\\
&+\frac{-3\,(-1+y)}{4}\,\Delta G(0,\lambda;0,y)\,G(x\lambda;x,y)\,H(0;y)+\\
&+\frac{-1+y}{4}\,\Delta G(0,x\lambda;0,y)\,G(\lambda;x,y)\,H(0;y)+\\
&+\frac{-(-1+y)^2}{4}\,\Delta G(0,x\lambda;0,y)\,G(x\lambda;x,y)\,H(0;y)-\\
&-2\,G(\lambda;0,y)\,G(\lambda;x,y)\,H(0,0;x_0)+G(\lambda;0,y)\,G(\lambda;x,y)\,H(0,0;y)+\\
&+\frac{1}{2}\,G(\lambda;0,y)\,G(\lambda;x,y)\,H(1,0;y)+\\
&+2\,(-1+y)\,G(\lambda;0,y)\,G(x\lambda;x,y)\,H(0,0;x_0)+\\
&+(1-y)\,G(\lambda;0,y)\,G(x\lambda;x,y)\,H(0,0;y)+\\
&+\frac{1-y}{2}\,G(\lambda;0,y)\,G(x\lambda;x,y)\,H(1,0;y)-\\
&-\frac{3}{2}\,G(\lambda;0,y)\,G(0,\lambda;x,y)\,H(0;x_0)+\frac{3}{4}\,G(\lambda;0,y)\,G(0,\lambda;x,y)\,H(0;y)+\\
&+\frac{3\,(-1+y)}{2}\,G(\lambda;0,y)\,G(0,x\lambda;x,y)\,H(0;x_0)+\\
&+\frac{-3\,(-1+y)}{4}\,G(\lambda;0,y)\,G(0,x\lambda;x,y)\,H(0;y)-\\
&-\frac{1}{2}\,G(\lambda;0,y)\,G(\lambda,0;x,y)\,H(0;x_0)+\frac{1}{4}\,G(\lambda;0,y)\,G(\lambda,0;x,y)\,H(0;y)-\\
&-\left(G(\lambda;0,y)\,G(x_0,\lambda;x,y)\,H(0;x_0)\right)+\frac{1}{2}\,G(\lambda;0,y)\,G(x_0,\lambda;x,y)\,H(0;y)+\\
&+(-1+y)\,G(\lambda;0,y)\,G(x_0,x\lambda;x,y)\,H(0;x_0)+\\
&+\frac{1-y}{2}\,G(\lambda;0,y)\,G(x_0,x\lambda;x,y)\,H(0;y)-\\
&-\left(G(\lambda;0,y)\,G(x_1,\lambda;x,y)\,H(0;x_0)\right)+\frac{1}{2}\,G(\lambda;0,y)\,G(x_1,\lambda;x,y)\,H(0;y)+\\
&+(-1+y)\,G(\lambda;0,y)\,G(x_1,x\lambda;x,y)\,H(0;x_0)+\\
&+\frac{1-y}{2}\,G(\lambda;0,y)\,G(x_1,x\lambda;x,y)\,H(0;y)\\
&+\frac{-1+y}{2}\,G(\lambda;0,y)\,G(x\lambda,0;x,y)\,H(0;x_0)+\\
&+\frac{1-y}{4}\,G(\lambda;0,y)\,G(x\lambda,0;x,y)\,H(0;y)-\\
&-\frac{3}{2}\,G(\lambda;x,y)\,G(\lambda,0;0,y)\,H(0;x_0)-\frac{1}{4}\,G(\lambda;x,y)\,G(\lambda,0;0,y)\,H(0;y)+\\
&+\frac{1}{2}\,G(\lambda;x,y)\,G(\lambda,\lambda;0,y)\,H(0;x_0)-\frac{1}{4}\,G(\lambda;x,y)\,G(\lambda,\lambda;0,y)\,H(0;y)+
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{-1+y}{2}\,G(\lambda;x,y)\,G(\lambda,x\lambda;0,y)\,H(0;x_0)+\\
&+\frac{-3\,(-1+y)}{4}\,G(\lambda;x,y)\,G(\lambda,x\lambda;0,y)\,H(0;y)-\\
&-\,(G(\lambda;x,y)\,G(x_0,\lambda;0,y)\,H(0;x_0))+\frac{1}{2}\,G(\lambda;x,y)\,G(x_0,\lambda;0,y)\,H(0;y)+\\
&+\frac{-1+y}{2}\,G(\lambda;x,y)\,G(x_0,x\lambda;0,y)\,H(0;y)-\\
&-\,(G(\lambda;x,y)\,G(x_1,\lambda;0,y)\,H(0;x_0))+\\
&+\frac{1}{2}\,G(\lambda;x,y)\,G(x_1,\lambda;0,y)\,H(0;y)+\\
&+\frac{-1+y}{2}\,G(\lambda;x,y)\,G(x_1,x\lambda;0,y)\,H(0;y)+\\
&+\frac{3\,(-1+y)}{2}\,G(x\lambda;x,y)\,G(\lambda,0;0,y)\,H(0;x_0)+\\
&+\frac{-1+y}{4}\,G(x\lambda;x,y)\,G(\lambda,0;0,y)\,H(0;y)+\\
&+\frac{1-y}{2}\,G(x\lambda;x,y)\,G(\lambda,\lambda;0,y)\,H(0;x_0)+\\
&+\frac{-1+y}{4}\,G(x\lambda;x,y)\,G(\lambda,\lambda;0,y)\,H(0;y)+\\
&+\frac{-(-1+y)^2}{2}\,G(x\lambda;x,y)\,G(\lambda,x\lambda;0,y)\,H(0;x_0)+\\
&+\frac{3\,(-1+y)^2}{4}\,G(x\lambda;x,y)\,G(\lambda,x\lambda;0,y)\,H(0;y)+\\
&+(-1+y)\,G(x\lambda;x,y)\,G(x_0,\lambda;0,y)\,H(0;x_0)+\\
&+\frac{1-y}{2}\,G(x\lambda;x,y)\,G(x_0,\lambda;0,y)\,H(0;y)+\\
&+\frac{-(-1+y)^2}{2}\,G(x\lambda;x,y)\,G(x_0,x\lambda;0,y)\,H(0;y)+\\
&+(-1+y)\,G(x\lambda;x,y)\,G(x_1,\lambda;0,y)\,H(0;x_0)+\\
&+\frac{1-y}{2}\,G(x\lambda;x,y)\,G(x_1,\lambda;0,y)\,H(0;y)+\\
&+\frac{-(-1+y)^2}{2}\,G(x\lambda;x,y)\,G(x_1,x\lambda;0,y)\,H(0;y)\,.
\end{aligned}
$$

The second momentum configuration is defined by,

$$F_1^b(p_1^2,p_2^2,p_3^2)\;=$$



$$= S_D^2(-p_3^2)^{-2\epsilon}\left(\sum_{i=-2,\ldots,1} f_1^i\left(\frac{p_1^2}{p_3^2},\frac{p_2^2}{p_3^2}\right)\epsilon^i+\mathcal{O}(\epsilon^2)\right),$$

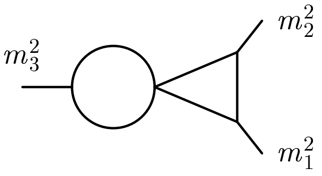


Figure 8: The two-loop master integral, $\mathrm{TB}(m_1^2, m_2^2, m_3^3)$.

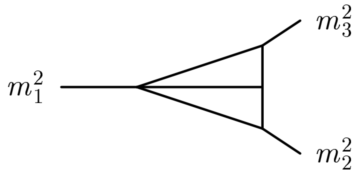
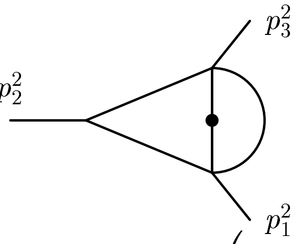


Figure 9: The two-loop master integral, $F_3(m_1^2, m_2^2, m_3^3)$.

$$F_2^b(p_1^2, p_2^2, p_3^2) =$$



$$= S_D^2 (-p_3^2)^{-1-2\epsilon} \frac{1}{\lambda} \left(\sum_{i=-1,\dots,1} f_2^i \left(\frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2} \right) \epsilon^i + \mathcal{O}(\epsilon^2) \right).$$

Expressions for $f_{1,2}^i$ in this momentum configuration are similarly lengthy to those for $F_{1,2}^a$ and can be obtained in computer readable form from the authors. In each case, we have checked that the leading contribution agrees with the results of ref. [54].

Note that solving the differential equation for the crossed triangle requires the functions $F_{1,2}^a(x,y)$ which are symmetric in x and y and in addition the functions $F_{1,2}^b(x,y)$ and $F_{1,2}^b(y,x)$. The functions $F_{1,2}^b(y,x)$ are, of course, in principle known. However, exchanging x and y puts them in a form that is not suited for further integration over x. To get them into a suitable form we therefore recalculate them directly from the differential equations.

6.2.2 Master integrals with five propagators

There are two master integrals with five propagators. The first is a product of one-loop integrals and the second is a genuine two-loop integral. The first, denoted by TB, is shown in figure 8 and is given by,

$$\mathrm{TB}(p_1^2, p_2^2, p_3^3) = \mathrm{BB}(p_i^2) F_0(p_1^2, p_2^2, p_3^3)\,. \tag{6.7}$$

The ϵ expansion for this integral is straightforwardly obtained from eq. (5.6) and we do not show it here.

The second irreducible two-loop diagram is denoted by F_3 and is shown in figure 9. It is finite in 4-dimensions and the leading contribution has been calculated in ref. [54].

The differential equation for F_3 is given by,

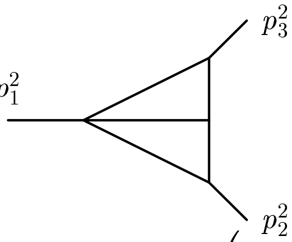
$$\begin{aligned}
\frac{\partial}{\partial m_1^2} F_3(m_1^2, m_2^2, m_3^2) \; = \; & \frac{(d-3)\,\left(-m_1^2 + m_2^2 + m_3^2\right)}{\Lambda^2} \mathrm{F}_3 \\
& - \frac{2\,(3\,d-10)\,(d-3)}{(d-4)\,\Lambda^2}\,\mathrm{F}_1(m_1^2, m_2^2, m_3^2) - \\
& - \frac{2\,(3\,d-10)\,(d-3)}{(d-4)\,\Lambda^2}\,\mathrm{F}_1(m_3^2, m_1^2, m_2^2) + \\
& + \frac{\left(-m_1^2 - m_2^2 + m_3^2\right)}{\Lambda^2}\,\mathrm{F}_2(m_1^2, m_2^2, m_3^2) - \\
& - \frac{\left(m_1^2 - m_2^2 + m_3^2\right)}{\Lambda^2}\,\mathrm{F}_2(m_3^2, m_1^2, m_2^2) + \frac{4\,(d-3)^2}{(d-4)\,\Lambda^2}\,\mathrm{TGL}(m_3^2, m_2^2)\,.
\end{aligned} \tag{6.8}$$

The solution of the homogeneous equation at $d = 4$ is given by $F_3^{hom} = \frac{1}{\lambda}$ and we therefore take the boundary condition at $\lambda = 0$ corresponding to $x \to x_0$, so that

$$\begin{aligned}
F_3(m_1^2, m_2^2, m_3^2)|_{m_1^2=(m_2-m_3)^2} \; = \; & -\frac{(3\,d-10)}{(d-4)\;(m_2-m_3)\;m_3}\,F_1({m_2}^2, {m_3}^2, (m_2-m_3)^2) - \\
& - \frac{(3\,d-10)}{(d-4)\;(m_2-m_3)\;m_3}\,F_1((m_2-m_3)^2, {m_2}^2, {m_3}^2) - \\
& - \frac{m_2}{(d-3)\;(m_2-m_3)}\,F_2({m_2}^2, {m_3}^2, (m_2-m_3)^2) - \\
& - \frac{m_2}{(d-3)\;m_3}\,F_2((m_2-m_3)^2, {m_2}^2, {m_3}^2) + \\
& + \frac{2\;(d-3)}{(d-4)\;(m_2-m_3)\;m_3}\,\mathrm{TGL}((m_2-m_3)^2, {m_3}^2)\,.
\end{aligned} \tag{6.9}$$

Once again, there are two distinct kinematic configurations depending on the position of the large scale p_3^2. For the first momentum configuration, the first two terms in the ϵ expansion are given by,

$$F_3^a(p_1^2, p_2^2, p_3^2) \; =$$



$$= S_D^2(-p_3^2)^{-1-2\epsilon}\frac{1}{\lambda}\left(\sum_{i=0,\ldots,1} f_3^i\left(\frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2}\right)\epsilon^i + \mathcal{O}(\epsilon^2)\right),$$

where,

$$\begin{aligned}
f_3^0 \; = \; & -6\,\zeta_3\,G(\lambda; x, y) + 6\,G(\lambda,\lambda,\lambda,0; x, y) + \\
& + \frac{-\pi^2}{3}\,G(\lambda; 0, y)\,G(\lambda; x, y) + 2\,G(\lambda; x, y)\,G(\lambda, 0, \lambda; 0, y) - \\
& - 2\,G(\lambda; x, y)\,G(\lambda,\lambda,0; 0, y) + \frac{\pi^2}{3}\,G(\lambda; x, y)\,H(0; y) +
\end{aligned}$$

$$
\begin{aligned}
& + 2\,G(\lambda;x,y)\,H(0,1,0;y) - 2\,G(\lambda;x,y)\,H(1,0,0;y) + \\
& + G(\lambda,0;x,y)\,H(0,0;y) + 6\,G(\lambda,\lambda,\lambda;x,y)\,H(0;x_0) - \\
& - 3\,G(\lambda,\lambda,\lambda;x,y)\,H(0;y) + (3-3\,y)\,G(\lambda,\lambda,x\lambda;x,y)\,H(0;y) - \\
& - \Delta G(0;0,y)\,G(\lambda;x,y)\,H(0,0;y) - \Delta G(0,\lambda;0,y)\,G(\lambda;x,y)\,H(0;y) - \\
& - G(\lambda;0,y)\,G(\lambda;x,y)\,H(0,0;y) - 2\,G(\lambda;0,y)\,G(\lambda;x,y)\,H(1,0;y) - \\
& - 2\,G(\lambda;x,y)\,G(\lambda,\lambda;0,y)\,H(0;x_0) + G(\lambda;x,y)\,G(\lambda,\lambda;0,y)\,H(0;y) + \\
& + 2\,\left(-1+y\right)\,G(\lambda;x,y)\,G(\lambda,x\lambda;0,y)\,H(0;y) \\
f_3^1 \;=\; & + \frac{-3\,\pi^4}{20}\,G(\lambda;x,y) - 3\,\zeta_3\,G(\lambda,0;x,y) + 6\,\zeta_3\,G(x_0,\lambda;x,y) + 6\,\zeta_3\,G(x_1,\lambda;x,y) + \\
& + \frac{-\left(\pi^2\,\left(-1+y\right)\right)}{2}\,G(\lambda,\lambda,x\lambda;x,y) + \frac{-\left(\pi^2\,\left(-1+y\right)\right)}{2}\,G(\lambda,x\lambda,\lambda;x,y) - \\
& - 3\,G(\lambda,0,\lambda,\lambda,0;x,y) - 9\,G(\lambda,\lambda,0,\lambda,0;x,y) - 12\,G(\lambda,\lambda,\lambda,0,0;x,y) - \\
& - 6\,G(\lambda,\lambda,x_0,\lambda,0;x,y) - 6\,G(\lambda,\lambda,x_1,\lambda,0;x,y) - 6\,G(x_0,\lambda,\lambda,\lambda,0;x,y) - \\
& - 6\,G(x_1,\lambda,\lambda,\lambda,0;x,y) + 3\,\zeta_3\,\Delta G(0;0,y)\,G(\lambda;x,y) + \\
& + \frac{\pi^2}{6}\,\Delta G(0,\lambda;0,y)\,G(\lambda;x,y) - 3\,\Delta G(0,\lambda,0,\lambda;0,y)\,G(\lambda;x,y) \\
& - 6\,\Delta G(0,\lambda,\lambda,0;0,y)\,G(\lambda;x,y) + \zeta_3\,G(\lambda;0,y)\,G(\lambda;x,y) + \\
& + \frac{\pi^2}{6}\,G(\lambda;0,y)\,G(\lambda,0;x,y) + \frac{\pi^2}{3}\,G(\lambda;0,y)\,G(x_0,\lambda;x,y) + \\
& + \frac{\pi^2}{3}\,G(\lambda;0,y)\,G(x_1,\lambda;x,y) + \frac{-\pi^2}{6}\,G(\lambda;x,y)\,G(\lambda,0;0,y) + \\
& + \frac{\pi^2\,\left(-1+y\right)}{6}\,G(\lambda;x,y)\,G(\lambda,x\lambda;0,y) + \frac{\pi^2}{3}\,G(\lambda;x,y)\,G(x_0,\lambda;0,y) + \\
& + \frac{\pi^2}{3}\,G(\lambda;x,y)\,G(x_1,\lambda;0,y) - 4\,G(\lambda;x,y)\,G(\lambda,0,0,\lambda;0,y) - \\
& - \left(G(\lambda;x,y)\,G(\lambda,0,\lambda,0;0,y)\right) - \\
& - 2\,G(\lambda;x,y)\,G(\lambda,\lambda,0,0;0,y) + (3-3\,y)\,G(\lambda;x,y)\,G(\lambda,\lambda,0,x\lambda;0,y) + \\
& + (3-3\,y)\,G(\lambda;x,y)\,G(\lambda,\lambda,x\lambda,0;0,y) + 4\,G(\lambda;x,y)\,G(\lambda,x_0,\lambda,0;0,y) + \\
& + 4\,G(\lambda;x,y)\,G(\lambda,x_1,\lambda,0;0,y) + (3-3\,y)\,G(\lambda;x,y)\,G(\lambda,x\lambda,\lambda,0;0,y) - \\
& - 2\,G(\lambda;x,y)\,G(x_0,\lambda,0,\lambda;0,y) - 4\,G(\lambda;x,y)\,G(x_0,\lambda,\lambda,0;0,y) - \\
& - 2\,G(\lambda;x,y)\,G(x_1,\lambda,0,\lambda;0,y) - \\
& - 4\,G(\lambda;x,y)\,G(x_1,\lambda,\lambda,0;0,y) + 8\,\zeta_3\,G(\lambda;x,y)\,H(0;y) + \\
& + 6\,\zeta_3\,G(\lambda;x,y)\,H(1;y) + \frac{-\pi^2}{3}\,G(\lambda;x,y)\,H(0,0;y) + \\
& + \frac{\pi^2}{3}\,G(\lambda;x,y)\,H(0,1;y) + \frac{\pi^2}{3}\,G(\lambda;x,y)\,H(1,0;y) + \\
& + 2\,G(\lambda;x,y)\,H(0,0,0,0;y) - \\
& - 2\,G(\lambda;x,y)\,H(0,0,1,0;y) - 2\,G(\lambda;x,y)\,H(0,1,0,0;y) + \\
& + 2\,G(\lambda;x,y)\,H(0,1,1,0;y) + 6\,G(\lambda;x,y)\,H(1,0,0,0;y) + \\
& + 2\,G(\lambda;x,y)\,H(1,0,1,0;y) - 2\,G(\lambda;x,y)\,H(1,1,0,0;y) +
\end{aligned}
$$

$$+3\,(-1+y)\;G(\lambda,0;0,y)\,G(\lambda,\lambda,x\lambda;x,y)+$$
$$+\,3\,(-1+y)\;G(\lambda,0;0,y)\,G(\lambda,x\lambda,\lambda;x,y)-$$
$$-\,\left(G(\lambda,0;x,y)\,G(\lambda,0,\lambda;0,y)\right)+$$
$$+\,G(\lambda,0;x,y)\,G(\lambda,\lambda,0;0,y)+\frac{-\pi^2}{6}\,G(\lambda,0;x,y)\,H(0;y)-$$
$$-\,3\,G(\lambda,0;x,y)\,H(0,0,0;y)\,(G(\lambda,0;x,y)\,H(0,1,0;y))+$$
$$+\,G(\lambda,0;x,y)\,H(1,0,0;y)-2\,G(x_0,\lambda;x,y)\,G(\lambda,0,\lambda;0,y)+$$
$$+\,2\,G(x_0,\lambda;x,y)\,G(\lambda,\lambda,0;0,y)+\frac{-\pi^2}{3}\,G(x_0,\lambda;x,y)\,H(0;y)-$$
$$-\,2\,G(x_0,\lambda;x,y)\,H(0,1,0;y)+2\,G(x_0,\lambda;x,y)\,H(1,0,0;y)-$$
$$-\,2\,G(x_1,\lambda;x,y)\,G(\lambda,0,\lambda;0,y)+2\,G(x_1,\lambda;x,y)\,G(\lambda,\lambda,0;0,y)+$$
$$+\,\frac{-\pi^2}{3}\,G(x_1,\lambda;x,y)\,H(0;y)-2\,G(x_1,\lambda;x,y)\,H(0,1,0;y)+$$
$$+\,2\,G(x_1,\lambda;x,y)\,H(1,0,0;y)-\frac{1}{2}\,G(\lambda,0,0;x,y)\,H(0,0;y)-$$
$$-\,12\,G(\lambda,\lambda,\lambda;x,y)\,H(0,0;x_0)+\frac{9}{2}\,G(\lambda,\lambda,\lambda;x,y)\,H(0,0;y)+$$
$$+\,\frac{9\,(-1+y)}{2}\,G(\lambda,\lambda,x\lambda;x,y)\,H(0,0;y)+$$
$$+\,(3-3\,y)\,G(\lambda,\lambda,x\lambda;x,y)\,H(1,0;y)+$$
$$+\,\frac{-3\,(-1+y)}{2}\,G(\lambda,x\lambda,\lambda;x,y)\,H(0,0;y)+$$
$$+\,(3-3\,y)\,G(\lambda,x\lambda,\lambda;x,y)\,H(1,0;y)+$$
$$+\,\frac{-3\,(-1+y)^2}{2}\,G(\lambda,x\lambda,x\lambda;x,y)\,H(0,0;y)-$$
$$-\,\left(G(x_0,\lambda,0;x,y)\,H(0,0;y)\right)-\left(G(x_1,\lambda,0;x,y)\,H(0,0;y)\right)-$$
$$-\,3\,G(\lambda,0,\lambda,\lambda;x,y)\,H(0;x_0)++\frac{3}{2}\,G(\lambda,0,\lambda,\lambda;x,y)\,H(0;y)+$$
$$+\,\frac{3\,(-1+y)}{2}\,G(\lambda,0,\lambda,x\lambda;x,y)\,H(0;y)-$$
$$-\,9\,G(\lambda,\lambda,0,\lambda;x,y)\,H(0;x_0)+\frac{9}{2}\,G(\lambda,\lambda,0,\lambda;x,y)\,H(0;y)+$$
$$+\,\frac{9\,(-1+y)}{2}\,G(\lambda,\lambda,0,x\lambda;x,y)\,H(0;y)-12\,G(\lambda,\lambda,\lambda,0;x,y)\,H(0;x_0)-$$
$$-6\,G(\lambda,\lambda,x_0,\lambda;x,y)\,H(0;x_0)+3\,G(\lambda,\lambda,x_0,\lambda;x,y)\,H(0;y)+$$
$$+\,3\,(-1+y)\;G(\lambda,\lambda,x_0,x\lambda;x,y)\,H(0;y)-6\,G(\lambda,\lambda,x_1,\lambda;x,y)\,H(0;x_0)+$$
$$+\,3\,G(\lambda,\lambda,x_1,\lambda;x,y)\,H(0;y)+3\,(-1+y)\;G(\lambda,\lambda,x_1,x\lambda;x,y)\,H(0;y)+$$
$$+\,\frac{3\,(-1+y)}{2}\,G(\lambda,\lambda,x\lambda,0;x,y)\,H(0;y)$$
$$+\,\frac{3\,(-1+y)}{2}\,G(\lambda,x\lambda,\lambda,0;x,y)\,H(0;y)-$$
$$-\,6\,G(x_0,\lambda,\lambda,\lambda;x,y)\,H(0;x_0)+3\,G(x_0,\lambda,\lambda,\lambda;x,y)\,H(0;y)+$$
$$+\,3\,(-1+y)\;G(x_0,\lambda,\lambda,x\lambda;x,y)\,H(0;y)-6\,G(x_1,\lambda,\lambda,\lambda;x,y)\,H(0;x_0)+$$

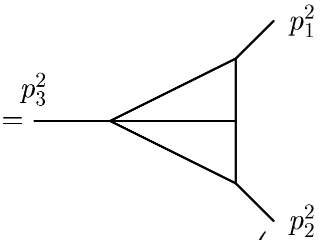
$$+\,3\,G(x_1,\lambda,\lambda,\lambda;x,y)\,H(0;y)+3\,(-1+y)\,G(x_1,\lambda,\lambda,x\lambda;x,y)\,H(0;y)+$$
$$+\,\frac{\pi^2}{6}\Delta G(0;0,y)\,G(\lambda;x,y)\,H(0;y)+3\,\Delta G(0;0,y)G(\lambda;x,y)\,H(0,0,0;y)+$$
$$+\,\Delta G(0;0,y)G(\lambda;x,y)\,H(0,1,0;y)-(\Delta G(0;0,y)G(\lambda;x,y)\,H(1,0,0;y))+$$
$$+\,\frac{1}{2}\,\Delta G(0;0,y)\,G(\lambda,0;x,y)\,H(0,0;y)+\Delta G(0;0,y)\,G(x_0,\lambda;x,y)\,H(0,0;y)+$$
$$+\,\Delta G(0;0,y)\,G(x_1,\lambda;x,y)\,H(0,0;y)+$$
$$+\,\frac{-3\,(-1+y)}{2}\,\Delta G(0;0,y)\,G(\lambda,\lambda,x\lambda;x,y)\,H(0;y)+$$
$$+\,\frac{-3\,(-1+y)}{2}\,\Delta G(0;0,y)\,G(\lambda,x\lambda,\lambda;x,y)\,H(0;y)-$$
$$-\,\frac{1}{2}\,\Delta G(0,0;0,y)\,G(\lambda;x,y)\,H(0,0;y)-$$
$$-\,(\Delta G(0,\lambda;0,y)\,G(\lambda;x,y)\,G(\lambda,0;0,y))+$$
$$+\,\frac{5}{2}\,\Delta G(0,\lambda;0,y)\,G(\lambda;x,y)\,H(0,0;y)+$$
$$+\,\Delta G(0,\lambda;0,y)\,G(\lambda;x,y)\,H(1,0;y)+\frac{1}{2}\,\Delta G(0,\lambda;0,y)\,G(\lambda,0;x,y)\,H(0;y)+$$
$$+,\,\Delta G(0,\lambda;0,y)\,G(x_0,\lambda;x,y)\,H(0;y)+$$
$$+\,\Delta G(0,\lambda;0,y)\,G(x_1,\lambda;x,y)\,H(0;y)+$$
$$+\,\frac{-1+y}{2}\,\Delta G(0,x\lambda;0,y)\,G(\lambda;x,y)\,H(0,0;y)+$$
$$+\,\frac{3}{2}\,\Delta G(0,0,\lambda;0,y)\,G(\lambda;x,y)\,H(0;y)-$$
$$-\,\frac{1}{2}\,\Delta G(0,\lambda,0;0,y)\,G(\lambda;x,y)\,H(0;y)-$$
$$-\,8\,\Delta G(0,\lambda,\lambda;0,y)\,G(\lambda;x,y)\,H(0;x_0)+$$
$$+\,4\,\Delta G(0,\lambda,\lambda;0,y)\,G(\lambda;x,y)\,H(0;y)+$$
$$+\,\frac{5\,(-1+y)}{2}\,\Delta G(0,\lambda,x\lambda;0,y)\,G(\lambda;x,y)\,H(0;y)+$$
$$+\,\Delta G(0,x_0,\lambda;0,y)\,G(\lambda;x,y)\,H(0;y)+$$
$$+\,\Delta G(0,x_1,\lambda;0,y)\,G(\lambda;x,y)\,H(0;y)+$$
$$+\,\frac{5\,(-1+y)}{2}\,\Delta G(0,x\lambda,\lambda;0,y)\,G(\lambda;x,y)\,H(0;y)$$
$$+\,\frac{\pi^2}{6}\,G(\lambda;0,y)\,G(\lambda;x,y)\,H(0;y)+$$
$$+\,\frac{-\pi^2}{3}\,G(\lambda;0,y)\,G(\lambda;x,y)\,H(1;y)+3\,G(\lambda;0,y)\,G(\lambda;x,y)\,H(0,0,0;y)+$$
$$+\,G(\lambda;0,y)\,G(\lambda;x,y)\,H(0,1,0;y)+3\,G(\lambda;0,y)\,G(\lambda;x,y)\,H(1,0,0;y)-$$
$$-\,2\,G(\lambda;0,y)\,G(\lambda;x,y)\,H(1,1,0;y)+\frac{1}{2}\,G(\lambda;0,y)\,G(\lambda,0;x,y)\,H(0,0;y)+$$
$$+\,G(\lambda;0,y)\,G(\lambda,0;x,y)\,H(1,0;y)+G(\lambda;0,y)\,G(x_0,\lambda;x,y)\,H(0,0;y)+$$
$$+\,2\,G(\lambda;0,y)\,G(x_0,\lambda;x,y)\,H(1,0;y)+G(\lambda;0,y)\,G(x_1,\lambda;x,y)\,H(0,0;y)+$$
$$+\,2\,G(\lambda;0,y)\,G(x_1,\lambda;x,y)\,H(1,0;y)+$$

$$
\begin{aligned}
&+3\,(-1+y)\;G(\lambda;0,y)\,G(\lambda,\lambda,x\lambda;x,y)\,H(0;x_0)+\\
&+\frac{-3\,(-1+y)}{2}\,G(\lambda;0,y)\,G(\lambda,\lambda,x\lambda;x,y)\,H(0;y)+\\
&+3\,(-1+y)\;G(\lambda;0,y)\,G(\lambda,x\lambda,\lambda;x,y)\,H(0;x_0)+\\
&+\frac{-3\,(-1+y)}{2}\,G(\lambda;0,y)\,G(\lambda,x\lambda,\lambda;x,y)\,H(0;y)+\\
&+G(\lambda;x,y)\,G(\lambda,0;0,y)^2+(1-y)\,G(\lambda;x,y)\,G(\lambda,0;0,y)\,G(\lambda,x\lambda;0,y)-\\
&-2\,G(\lambda;x,y)\,G(\lambda,0;0,y)\,G(x_0,\lambda;0,y)-\\
&-2\,G(\lambda;x,y)\,G(\lambda,0;0,y)\,G(x_1,\lambda;0,y)-\frac{1}{2}\,G(\lambda;x,y)\,G(\lambda,0;0,y)\,H(0,0;y)-\\
&-\left(G(\lambda;x,y)\,G(\lambda,0;0,y)\,H(1,0;y)\right)+4\,G(\lambda;x,y)\,G(\lambda,\lambda;0,y)\,H(0,0;x_0)-\\
&-\frac{3}{2}\,G(\lambda;x,y)\,G(\lambda,\lambda;0,y)\,H(0,0;y)+\\
&+\frac{-7\,(-1+y)}{2}\,G(\lambda;x,y)\,G(\lambda,x\lambda;0,y)\,H(0,0;y)+\\
&+(-1+y)\,G(\lambda;x,y)\,G(\lambda,x\lambda;0,y)\,H(1,0;y)\\
&+G(\lambda;x,y)\,G(x_0,\lambda;0,y)\,H(0,0;y)+2\,G(\lambda;x,y)\,G(x_0,\lambda;0,y)\,H(1,0;y)+\\
&+(-1+y)\,G(\lambda;x,y)\,G(x_0,x\lambda;0,y)\,H(0,0;y)\\
&+G(\lambda;x,y)\,G(x_1,\lambda;0,y)\,H(0,0;y)+2\,G(\lambda;x,y)\,G(x_1,\lambda;0,y)\,H(1,0;y)+\\
&+(-1+y)\,G(\lambda;x,y)\,G(x_1,x\lambda;0,y)\,H(0,0;y)-\\
&-\left(G(\lambda;x,y)\,G(\lambda,0,\lambda;0,y)\,H(0;x_0)\right)-\\
&-\frac{3}{2}\,G(\lambda;x,y)\,G(\lambda,0,\lambda;0,y)\,H(0;y)+\\
&+(1-y)\,G(\lambda;x,y)\,G(\lambda,0,x\lambda;0,y)\,H(0;y)+\\
&+3\,G(\lambda;x,y)\,G(\lambda,\lambda,0;0,y)\,H(0;x_0)+\\
&+\frac{1}{2}\,G(\lambda;x,y)\,G(\lambda,\lambda,0;0,y)\,H(0;y)\\
&+(5-5\,y)\,G(\lambda;x,y)\,G(\lambda,\lambda,x\lambda;0,y)\,H(0;x_0)+\\
&+\frac{5\,(-1+y)}{2}\,G(\lambda;x,y)\,G(\lambda,\lambda,x\lambda;0,y)\,H(0;y)+\\
&+2\,G(\lambda;x,y)\,G(\lambda,x_0,\lambda;0,y)\,H(0;x_0)-\\
&-\left(G(\lambda;x,y)\,G(\lambda,x_0,\lambda;0,y)\,H(0;y)\right)+\\
&+(2-2\,y)\,G(\lambda;x,y)\,G(\lambda,x_0,x\lambda;0,y)\,H(0;y)+\\
&+2\,G(\lambda;x,y)\,G(\lambda,x_1,\lambda;0,y)\,H(0;x_0)-\\
&-\left(G(\lambda;x,y)\,G(\lambda,x_1,\lambda;0,y)\,H(0;y)\right)+\\
&+(2-2\,y)\,G(\lambda;x,y)\,G(\lambda,x_1,x\lambda;0,y)\,H(0;y)+\\
&+(-1+y)\,G(\lambda;x,y)\,G(\lambda,x\lambda,0;0,y)\,H(0;y)+\\
&+(5-5\,y)\,G(\lambda;x,y)\,G(\lambda,x\lambda,\lambda;0,y)\,H(0;x_0)+\\
&+\frac{5\,(-1+y)}{2}\,G(\lambda;x,y)\,G(\lambda,x\lambda,\lambda;0,y)\,H(0;y)+\\
&+4\,(-1+y)^2\,G(\lambda;x,y)\,G(\lambda,x\lambda,x\lambda;0,y)\,H(0;y)+
\end{aligned}
$$

$$
\begin{aligned}
&+ G(\lambda;x,y)\, G(x_0,0,\lambda;0,y)\, H(0;y) - 8\, G(\lambda;x,y)\, G(x_0,\lambda,\lambda;0,y)\, H(0;x_0) + \\
&+ 4\, G(\lambda;x,y)\, G(x_0,\lambda,\lambda;0,y)\, H(0;y) + \\
&+ 2\,(-1+y)\; G(\lambda;x,y)\, G(x_0,\lambda,x\lambda;0,y)\, H(0;y) + \\
&+ 2\,(-1+y)\; G(\lambda;x,y)\, G(x_0,x\lambda,\lambda;0,y)\, H(0;y) + \\
&+ G(\lambda;x,y)\, G(x_1,0,\lambda;0,y)\, H(0;y) - \\
&- 8G(\lambda;x,y)G(x_1,\lambda,\lambda;0,y)\, H(0;x_0) + 4G(\lambda;x,y)G(x_1,\lambda,\lambda;0,y)\, H(0;y) + \\
&+ 2\,(-1+y)\; G(\lambda;x,y)\, G(x_1,\lambda,x\lambda;0,y)\, H(0;y) + \\
&+ 2\,(-1+y)\; G(\lambda;x,y)\, G(x_1,x\lambda,\lambda;0,y)\, H(0;y) + \\
&+ G(\lambda,0;x,y)\, G(\lambda,\lambda;0,y)\, H(0;x_0) - \frac{1}{2}\, G(\lambda,0;x,y)\, G(\lambda,\lambda;0,y)\, H(0;y) + \\
&+ (1-y)\, G(\lambda,0;x,y)\, G(\lambda,x\lambda;0,y)\, H(0;y) + \\
&+ 2\, G(\lambda,\lambda;0,y)\, G(x_0,\lambda;x,y)\, H(0;x_0) - \\
&- \;(G(\lambda,\lambda;0,y)\, G(x_0,\lambda;x,y)\, H(0;y)) + \\
&+ 2\, G(\lambda,\lambda;0,y)\, G(x_1,\lambda;x,y)\, H(0;x_0) - \\
&- \;(G(\lambda,\lambda;0,y)\, G(x_1,\lambda;x,y)\, H(0;y)) \\
&+ (2-2\,y)\, G(\lambda,x\lambda;0,y)\, G(x_0,\lambda;x,y)\, H(0;y) + \\
&+ (2-2\,y)\, G(\lambda,x\lambda;0,y)\, G(x_1,\lambda;x,y)\, H(0;y)
\end{aligned}
$$

The other momentum configuration required is defined by,

$$F_3^b(p_1^2,p_2^2,p_3^2) =$$



$$= S_D^2(-p_3^2)^{-1-2\epsilon}\frac{1}{\lambda}\left(\sum_{i=0,\ldots,1} f_3^i\left(\frac{p_1^2}{p_3^2},\frac{p_2^2}{p_3^2}\right)\epsilon^i + \mathcal{O}(\epsilon^2)\right).$$

Expressions for f_3^i in this momentum configuration are similarly lengthy to those for F_3^a and can be obtained in computer readable form from the authors. In each case, we have checked that the finite contribution agrees with the results of ref. [54].

6.2.3 Master integrals with six propagators

The only MI with 6 propagators is the crossed triangle which is denoted by F_4 and is shown in figure 10.

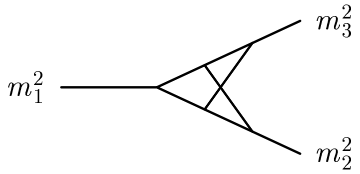


Figure 10: The two-loop master integral, $F_4(m_1^2, m_2^2, m_3^3)$.

This integral is fully symmetric in all three legs. and satisfies the differential equation,

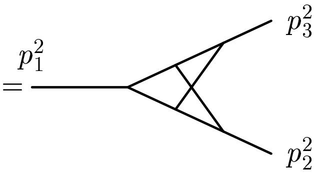
$$\begin{aligned}
\frac{\partial}{\partial m_1^2} F_4 = & \frac{(6-d)\left(-{m_1}^2+{m_2}^2+{m_3}^2\right)}{\Lambda^2} F_4 - \frac{8}{\Lambda^2} F_2({m_1}^2,{m_2}^2,{m_3}^2) + \\
& + \frac{4\left({m_1}^2+{m_2}^2-{m_3}^2\right)}{\Lambda^2\,{m_1}^2} F_2({m_2}^2,{m_3}^2,{m_1}^2) - \\
& - \frac{4\left(-{m_1}^2+{m_2}^2-{m_3}^2\right)}{\Lambda^2\,{m_1}^2} F_2({m_3}^2,{m_2}^2,{m_1}^2) - \\
& - \frac{4\,(\mathrm{d}-4)\,F_3({m_1}^2,{m_2}^2,{m_3}^2)}{\Lambda^2} + \\
& + \frac{2\,(\mathrm{d}-4)\left({m_1}^2+{m_2}^2-{m_3}^2\right)}{\Lambda^2\,{m_1}^2} F_3({m_2}^2,{m_3}^2,{m_1}^2) - \\
& - \frac{2\,(\mathrm{d}-4)\left(-{m_1}^2+{m_2}^2-{m_3}^2\right)}{\Lambda^2\,{m_1}^2} F_3({m_3}^2,{m_2}^2,{m_1}^2)\,. \qquad (6.10)
\end{aligned}$$

The homogeneous solution at $d = 4$ is $F_4^{hom} = \lambda^{-2}$ while the boundary condition at $\lambda = 0$ corresponding to $x \to x_0$ is given by,

$$\begin{aligned}
F_4(m_1^2, m_2^2, m_3^2)|_{m_1^2=(m_2-m_3)^2} = & \frac{4\,(m_2-m_3)}{(d-6)\,m_1^2\,m_3} F_2(m_2^2, m_3^2, (m_2-m_3)^2) - \\
& - \frac{4}{(d-6)\,m_2\,m_3} F_2((m_2-m_3)^2, m_2^2, m_3^2) - \\
& - \frac{4\,(m_2-m_3)}{(d-6)\,m_1^2\,m_2} F_2(m_3^2, m_2^2, (m_2-m_3)^2) + \\
& + \frac{2\,(d-4)\,(m_2-m_3)}{(d-6)\,m_1^2\,m_3} F_3(m_2^2, m_3^2, (m_2-m_3)^2) - \\
& - \frac{2\,(d-4)}{(d-6)\,m_2\,m_3} F_3((m_2-m_3)^2, m_2^2, m_3^2) - \\
& - \frac{2\,(d-4)\,(m_2-m_3)}{(d-6)\,m_1^2\,m_2} F_3(m_3^2, m_2^2, (m_2-m_3)^2)\,. \qquad (6.11)
\end{aligned}$$

We find that the first two terms of the ϵ-expansion are given by,

$$F_4(p_1^2, p_2^2, p_3^2) =$$



$$= S_D^2(-p_3^2)^{-2\epsilon}\left(\sum_{i=0,\ldots,1} f_1^i\left(\frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2}\right)\epsilon^i + \mathcal{O}(\epsilon^2)\right),$$

where,

$$
\begin{aligned}
f_4^0(x,y) \;=\; & + \frac{-8\,\pi^2}{3}\,G(\lambda,\lambda;x,y) + \frac{-4\,\pi^2\,(-1+y)}{3}\,G(\lambda,x\lambda;x,y) + \\
& + \frac{-4\,\pi^2\,(-1+y)}{3}\,G(x\lambda,\lambda;x,y) \\
& -8\,G(\lambda,0,\lambda,0;x,y) - 16\,G(\lambda,\lambda,0,0;x,y) + \\
& + 16\,G(\lambda,0;0,x_0)\,G(\lambda,\lambda;x,y) + \\
& +8\,(-1+y)\,G(\lambda,0;0,y)\,G(\lambda,x\lambda;x,y) + 8\,(-1+y)\,G(\lambda,0;0,y)\,G(x\lambda,\lambda;x,y) - \\
& - 16\,G(\lambda,\lambda;x,y)\,H(0,0;x_0) - 4\,G(\lambda,\lambda;x,y)\,H(0,0;y) - \\
& - 16\,G(\lambda,\lambda;x,y)\,H(1,0;x_0) + (4-4\,y)\,G(\lambda,x\lambda;x,y)\,H(0,0;y) + \\
& + (8-8\,y)\,G(\lambda,x\lambda;x,y)\,H(1,0;y) + (4-4\,y)\,G(x\lambda,\lambda;x,y)\,H(0,0;y) + \\
& + (8-8\,y)\,G(x\lambda,\lambda;x,y)\,H(1,0;y) - \\
& - 4\,(-1+y)^2\,G(x\lambda,x\lambda;x,y)\,H(0,0;y) - 8\,G(\lambda,0,\lambda;x,y)\,H(0;x_0) + \\
& + 4\,G(\lambda,0,\lambda;x,y)\,H(0;y) + 4\,(-1+y)\,G(\lambda,0,x\lambda;x,y)\,H(0;y) + \\
& - 16\,G(\lambda,\lambda,0;x,y)\,H(0;x_0) + 8\,G(\lambda,\lambda,0;x,y)\,H(0;y) + \\
& + 4\,(-1+y)\,G(\lambda,x\lambda,0;x,y)\,H(0;y) + 4\,(-1+y)\,G(x\lambda,\lambda,0;x,y)\,H(0;y) - \\
& - 8\,\Delta G(0;0,x_0)\,G(\lambda,\lambda;x,y)\,H(0;x_0) + (4-4\,y)\,\Delta G(0;0,y)\,G(\lambda,x\lambda;x,y)\,H(0;y) + \\
& + (4-4\,y)\,\Delta G(0;0,y)\,G(x\lambda,\lambda;x,y)\,H(0;y) - 8\,G(\lambda;0,x_0)\,G(\lambda,\lambda;x,y)\,H(0;x_0) + \\
& + 16\,G(\lambda;0,x_0)\,G(\lambda,\lambda;x,y)\,H(0;y) + \\
& +8\,(-1+y)\,G(\lambda;0,y)\,G(\lambda,x\lambda;x,y)\,H(0;x_0) + \\
& + (4-4\,y)\,G(\lambda;0,y)\,G(\lambda,x\lambda;x,y)\,H(0;y) + \\
& + 8\,(-1+y)\,G(\lambda;0,y)\,G(x\lambda,\lambda;x,y)\,H(0;x_0) + \\
& + (4-4\,y)\,G(\lambda;0,y)\,G(x\lambda,\lambda;x,y)\,H(0;y) \\
f_4^1(x,y) \;=\; & +112\,\zeta_3\,G(\lambda,\lambda;x,y) + 8\,(-1+y)\,\zeta_3\,G(\lambda,x\lambda;x,y) + \\
& + 8\,(-1+y)\,\zeta_3\,G(x\lambda,\lambda;x,y) + 32\,(-1+y)^2\,\zeta_3\,G(x\lambda,x\lambda;x,y) + \\
& + \frac{4\,\pi^2}{3}\,G(\lambda,0,\lambda;x,y) + \frac{4\,\pi^2\,(-1+y)}{3}\,G(\lambda,0,x\lambda;x,y) + \\
& + \frac{4\,\pi^2}{3}\,G(\lambda,\lambda,0;x,y) + \frac{8\,\pi^2}{3}\,G(\lambda,x_0,\lambda;x,y) + \frac{4\,\pi^2\,(-1+y)}{3}\,G(\lambda,x_0,x\lambda;x,y) + \\
& + \frac{8\,\pi^2}{3}\,G(\lambda,x_1,\lambda;x,y) + \frac{4\,\pi^2\,(-1+y)}{3}\,G(\lambda,x_1,x\lambda;x,y) + \\
& + \frac{4\,\pi^2\,(-1+y)}{3}\,G(\lambda,x\lambda,0;x,y) + \frac{-8\,\pi^2}{3}\,G(x_0,\lambda,\lambda;x,y) + \\
& + \frac{-4\,\pi^2\,(-1+y)}{3}\,G(x_0,\lambda,x\lambda;x,y) + \frac{-4\,\pi^2\,(-1+y)}{3}\,G(x_0,x\lambda,\lambda;x,y) + \\
& + \frac{-8\,\pi^2}{3}\,G(x_1,\lambda,\lambda;x,y) + \frac{-4\,\pi^2\,(-1+y)}{3}\,G(x_1,\lambda,x\lambda;x,y) + \\
& + \frac{-4\,\pi^2\,(-1+y)}{3}\,G(x_1,x\lambda,\lambda;x,y) + \frac{8\,\pi^2\,(-1+y)}{3}\,G(x\lambda,0,\lambda;x,y) +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4\,\pi^2\,(-1+y)}{3}\,G(x\lambda,\lambda,0;x,y)+\frac{4\,\pi^2\,(-1+y)}{3}\,G(x\lambda,x_0,\lambda;x,y)+\\
&+\frac{4\,\pi^2\,(-1+y)}{3}\,G(x\lambda,x_1,\lambda;x,y)+\frac{-4\,\pi^2\,(-1+y)^2}{3}\,G(x\lambda,x\lambda,0;x,y)+\\
&+16\,G(\lambda,0,0,\lambda,0;x,y)+24\,G(\lambda,0,\lambda,0,0;x,y)+\\
&+8\,G(\lambda,0,x_0,\lambda,0;x,y)+8\,G(\lambda,0,x_1,\lambda,0;x,y)+\\
&+48\,G(\lambda,\lambda,0,0,0;x,y)-72\,G(\lambda,\lambda,\lambda,\lambda,0;x,y)+\\
&+8\,G(\lambda,x_0,0,\lambda,0;x,y)+16\,G(\lambda,x_0,\lambda,0,0;x,y)+\\
&+8\,G(\lambda,x_1,0,\lambda,0;x,y)+16\,G(\lambda,x_1,\lambda,0,0;x,y)+\\
&+24\,(-1+y)^2\,G(\lambda,x\lambda,x\lambda,\lambda,0;x,y)-8\,G(x_0,\lambda,0,\lambda,0;x,y)-\\
&-16\,G(x_0,\lambda,\lambda,0,0;x,y)-8\,G(x_1,\lambda,0,\lambda,0;x,y)-\\
&-16\,G(x_1,\lambda,\lambda,0,0;x,y)+24\,(-1+y)^2\,G(x\lambda,\lambda,x\lambda,\lambda,0;x,y)+\\
&+24\,(-1+y)^2\,G(x\lambda,x\lambda,\lambda,\lambda,0;x,y)+\frac{-2\,\pi^2}{3}\,\Delta G(0;0,x_0)\,G(\lambda,\lambda;x,y)+\\
&+\frac{2\,\pi^2\,(-1+y)^2}{3}\,\Delta G(0;0,x_0)\,G(x\lambda,x\lambda;x,y)+\\
&+\frac{-4\,\pi^2\,(-1+y)}{3}\,\Delta G(0;0,y)\,G(\lambda,x\lambda;x,y)+\\
&+\frac{-4\,\pi^2\,(-1+y)}{3}\,\Delta G(0;0,y)\,G(x\lambda,\lambda;x,y)-12\,\Delta G(0,\lambda,0;0,x_0)\,G(\lambda,\lambda;x,y)+\\
&+12\,(-1+y)^2\,\Delta G(0,\lambda,0;0,x_0)\,G(x\lambda,x\lambda;x,y)\\
&-16\,(-1+y)\,\Delta G(0,\lambda,0;0,y)\,G(\lambda,x\lambda;x,y)-\\
&-16\,(-1+y)\,\Delta G(0,\lambda,0;0,y)\,G(x\lambda,\lambda;x,y)\\
&+\frac{-2\,\pi^2}{3}\,G(\lambda;0,x_0)\,G(\lambda,\lambda;x,y)+\frac{2\,\pi^2\,(-1+y)^2}{3}\,G(\lambda;0,x_0)\,G(x\lambda,x\lambda;x,y)+\\
&+\frac{8\,\pi^2}{3}\,G(\lambda;0,y)\,G(\lambda,\lambda;x,y)-8\,G(\lambda,0;0,x_0)\,G(\lambda,0,\lambda;x,y)-\\
&-8\,G(\lambda,0;0,x_0)\,G(\lambda,\lambda,0;x,y)-16\,G(\lambda,0;0,x_0)\,G(\lambda,x_0,\lambda;x,y)-\\
&-16\,G(\lambda,0;0,x_0)\,G(\lambda,x_1,\lambda;x,y)+\\
&+16\,G(\lambda,0;0,x_0)\,G(x_0,\lambda,\lambda;x,y)+\\
&+16\,G(\lambda,0;0,x_0)\,G(x_1,\lambda,\lambda;x,y)+\\
&+8\,(-1+y)^2\,G(\lambda,0;0,x_0)\,G(x\lambda,x\lambda,0;x,y)+\\
&+(8-8\,y)\,G(\lambda,0;0,y)\,G(\lambda,0,x\lambda;x,y)+\\
&+(8-8\,y)\,G(\lambda,0;0,y)\,G(\lambda,x_0,x\lambda;x,y)+(8-8\,y)\,G(\lambda,0;0,y)\,G(\lambda,x_1,x\lambda;x,y)+\\
&+(8-8\,y)\,G(\lambda,0;0,y)\,G(\lambda,x\lambda,0;x,y)+8\,(-1+y)\,G(\lambda,0;0,y)\,G(x_0,\lambda,x\lambda;x,y)+\\
&+8\,(-1+y)\,G(\lambda,0;0,y)\,G(x_0,x\lambda,\lambda;x,y)+\\
&+8\,(-1+y)\,G(\lambda,0;0,y)\,G(x_1,\lambda,x\lambda;x,y)+\\
&+8\,(-1+y)\,G(\lambda,0;0,y)\,G(x_1,x\lambda,\lambda;x,y)\\
&-16\,(-1+y)\,G(\lambda,0;0,y)\,G(x\lambda,0,\lambda;x,y)+
\end{aligned}
$$

$$
\begin{aligned}
&+(8-8\,y)\,G(\lambda,0;0,y)\,G(x\lambda,\lambda,0;x,y)+(8-8\,y)\,G(\lambda,0;0,y)\,G(x\lambda,x_0,\lambda;x,y)+\\
&+(8-8\,y)\,G(\lambda,0;0,y)\,G(x\lambda,x_1,\lambda;x,y)-16\,G(\lambda,\lambda;x,y)\,G(\lambda,0,0;0,x_0)-\\
&-16\,G(\lambda,\lambda;x,y)\,G(\lambda,0,\lambda;0,y)-12\,G(\lambda,\lambda;x,y)\,G(\lambda,\lambda,0;0,x_0)+\\
&+16\,G(\lambda,\lambda;x,y)\,G(\lambda,\lambda,0;0,y)-8\,G(\lambda,\lambda;x,y)\,G(x_0,\lambda,0;0,x_0)-\\
&-8\,G(\lambda,\lambda;x,y)\,G(x_1,\lambda,0;0,x_0)+\frac{4\,\pi^2}{3}\,G(\lambda,\lambda;x,y)\,H(0;x_0)+\\
&+\frac{-8\,\pi^2}{3}\,G(\lambda,\lambda;x,y)\,H(0;y)+\frac{-4\,\pi^2}{3}\,G(\lambda,\lambda;x,y)\,H(1;x_0)+\\
&+48\,G(\lambda,\lambda;x,y)\,H(0,0,0;x_0)+12\,G(\lambda,\lambda;x,y)\,H(0,0,0;y)+\\
&+8\,G(\lambda,\lambda;x,y)\,H(0,1,0;x_0)-16\,G(\lambda,\lambda;x,y)\,H(0,1,0;y)+\\
&+8\,G(\lambda,\lambda;x,y)\,H(1,0,0;x_0)+16\,G(\lambda,\lambda;x,y)\,H(1,0,0;y)-\\
&-8\,G(\lambda,\lambda;x,y)\,H(1,1,0;x_0)-16\,(-1+y)\,G(\lambda,x\lambda;x,y)\,G(\lambda,0,0;0,y)+\\
&+8\,(-1+y)^2\,G(\lambda,x\lambda;x,y)\,G(\lambda,0,x\lambda;0,y)+\\
&+8\,(-1+y)^2\,G(\lambda,x\lambda;x,y)\,G(\lambda,x\lambda,0;0,y)-\\
&-16\,(-1+y)\,G(\lambda,x\lambda;x,y)\,G(x_0,\lambda,0;0,y)-\\
&-16\,(-1+y)\,G(\lambda,x\lambda;x,y)\,G(x_1,\lambda,0;0,y)+\\
&+\frac{4\,\pi^2\,(-1+y)}{3}\,G(\lambda,x\lambda;x,y)\,H(0;y)+\\
&+\frac{-8\,\pi^2\,(-1+y)}{3}\,G(\lambda,x\lambda;x,y)\,H(1;y)+\\
&+12\,(-1+y)\,G(\lambda,x\lambda;x,y)\,H(0,0,0;y)+\\
&+8\,(-1+y)\,G(\lambda,x\lambda;x,y)\,H(0,1,0;y)+\\
&+24\,(-1+y)\,G(\lambda,x\lambda;x,y)\,H(1,0,0;y)-\\
&-16\,(-1+y)\,G(\lambda,x\lambda;x,y)\,H(1,1,0;y)-\\
&-16\,(-1+y)\,G(x\lambda,\lambda;x,y)\,G(\lambda,0,0;0,y)+\\
&+8\,(-1+y)^2\,G(x\lambda,\lambda;x,y)\,G(\lambda,0,x\lambda;0,y)+\\
&+8\,(-1+y)^2\,G(x\lambda,\lambda;x,y)\,G(\lambda,x\lambda,0;0,y)-\\
&-16\,(-1+y)\,G(x\lambda,\lambda;x,y)\,G(x_0,\lambda,0;0,y)-\\
&-16\,(-1+y)\,G(x\lambda,\lambda;x,y)\,G(x_1,\lambda,0;0,y)+\\
&+\frac{4\,\pi^2\,(-1+y)}{3}\,G(x\lambda,\lambda;x,y)\,H(0;y)+\\
&+\frac{-8\,\pi^2\,(-1+y)}{3}G(x\lambda,\lambda;x,y)\,H(1;y)+\\
&+12\,(-1+y)\,G(x\lambda,\lambda;x,y)H(0,0,0;y)+\\
&+8\,(-1+y)\,G(x\lambda,\lambda;x,y)\,H(0,1,0;y)+\\
&+24\,(-1+y)\,G(x\lambda,\lambda;x,y)\,H(1,0,0;y)-\\
&-16\,(-1+y)\,G(x\lambda,\lambda;x,y)\,H(1,1,0;y)+\\
&+16\,(-1+y)^2\,G(x\lambda,x\lambda;x,y)\,G(\lambda,0,0;0,x_0)+
\end{aligned}
$$

$$
\begin{aligned}
&+12\,(-1+y)^2\,G(x\lambda,x\lambda;x,y)\,G(\lambda,\lambda,0;0,x_0)+\\
&+8\,(-1+y)^2\,G(x\lambda,x\lambda;x,y)\,G(x_0,\lambda,0;0,x_0)+\\
&+8\,(-1+y)^2\,G(x\lambda,x\lambda;x,y)\,G(x_1,\lambda,0;0,x_0)+\\
&+\frac{-4\,\pi^2\,(-1+y)^2}{3}\,G(x\lambda,x\lambda;x,y)\,H(0;x_0)+\\
&+\frac{4\,\pi^2\,(-1+y)^2}{3}\,G(x\lambda,x\lambda;x,y)\,H(1;x_0)+\\
&+12\,(-1+y)^2\,G(x\lambda,x\lambda;x,y)\,H(0,0,0;y)-\\
&-8\,(-1+y)^2\,G(x\lambda,x\lambda;x,y)\,H(0,1,0;x_0)-\\
&-8\,(-1+y)^2\,G(x\lambda,x\lambda;x,y)\,H(1,0,0;x_0)+\\
&+8\,(-1+y)^2\,G(x\lambda,x\lambda;x,y)\,H(1,1,0;x_0)+\\
&+24\,G(\lambda,0,\lambda;x,y)\,H(0,0;x_0)-4\,G(\lambda,0,\lambda;x,y)\,H(0,0;y)+\\
&+8\,G(\lambda,0,\lambda;x,y)\,H(1,0;x_0)+(4-4\,y)\,G(\lambda,0,x\lambda;x,y)\,H(0,0;y)+\\
&+8\,(-1+y)\,G(\lambda,0,x\lambda;x,y)\,H(1,0;y)+48\,G(\lambda,\lambda,0;x,y)\,H(0,0;x_0)-\\
&-20\,G(\lambda,\lambda,0;x,y)\,H(0,0;y)+8\,G(\lambda,\lambda,0;x,y)\,H(1,0;x_0)\\
&+16\,G(\lambda,x_0,\lambda;x,y)\,H(0,0;x_0)+4\,G(\lambda,x_0,\lambda;x,y)\,H(0,0;y)+\\
&+16\,G(\lambda,x_0,\lambda;x,y)\,H(1,0;x_0)+4\,(-1+y)\,G(\lambda,x_0,x\lambda;x,y)\,H(0,0;y)+\\
&+8\,(-1+y)\,G(\lambda,x_0,x\lambda;x,y)\,H(1,0;y)+16\,G(\lambda,x_1,\lambda;x,y)\,H(0,0;x_0)+\\
&+4\,G(\lambda,x_1,\lambda;x,y)\,H(0,0;y)+16\,G(\lambda,x_1,\lambda;x,y)\,H(1,0;x_0)+\\
&+4\,(-1+y)\,G(\lambda,x_1,x\lambda;x,y)\,H(0,0;y)+8\,(-1+y)\,G(\lambda,x_1,x\lambda;x,y)\,H(1,0;y)+\\
&+(8-8\,y)\,G(\lambda,x\lambda,0;x,y)\,H(0,0;y)+8\,(-1+y)\,G(\lambda,x\lambda,0;x,y)\,H(1,0;y)-\\
&-16\,G(x_0,\lambda,\lambda;x,y)\,H(0,0;x_0)-4\,G(x_0,\lambda,\lambda;x,y)\,H(0,0;y)-\\
&-16\,G(x_0,\lambda,\lambda;x,y)\,H(1,0;x_0)+(4-4\,y)\,G(x_0,\lambda,x\lambda;x,y)\,H(0,0;y)+\\
&+(8-8\,y)\,G(x_0,\lambda,x\lambda;x,y)\,H(1,0;y)+(4-4\,y)\,G(x_0,x\lambda,\lambda;x,y)\,H(0,0;y)+\\
&+(8-8\,y)\,G(x_0,x\lambda,\lambda;x,y)\,H(1,0;y)-4(-1+y)^2G(x_0,x\lambda,x\lambda;x,y)H(0,0;y)-\\
&-16\,G(x_1,\lambda,\lambda;x,y)\,H(0,0;x_0)-4\,G(x_1,\lambda,\lambda;x,y)\,H(0,0;y)-\\
&-16\,G(x_1,\lambda,\lambda;x,y)\,H(1,0;x_0)+(4-4\,y)\,G(x_1,\lambda,x\lambda;x,y)\,H(0,0;y)+\\
&+(8-8\,y)\,G(x_1,\lambda,x\lambda;x,y)\,H(1,0;y)+(4-4\,y)\,G(x_1,x\lambda,\lambda;x,y)\,H(0,0;y)+\\
&+(8-8\,y)\,G(x_1,x\lambda,\lambda;x,y)\,H(1,0;y)-4\,(-1+y)^2\,G(x_1,x\lambda,x\lambda;x,y)\,H(0,0;y)+\\
&+8\,(-1+y)\,G(x\lambda,0,\lambda;x,y)\,H(0,0;y)+16\,(-1+y)\,G(x\lambda,0,\lambda;x,y)\,H(1,0;y)+\\
&+8\,(-1+y)^2\,G(x\lambda,0,x\lambda;x,y)\,H(0,0;y)+(8-8\,y)\,G(x\lambda,\lambda,0;x,y)\,H(0,0;y)+\\
&+8\,(-1+y)\,G(x\lambda,\lambda,0;x,y)\,H(1,0;y)+\\
&+4\,(-1+y)\,G(x\lambda,x_0,\lambda;x,y)\,H(0,0;y)+\\
&+8\,(-1+y)\,G(x\lambda,x_0,\lambda;x,y)\,H(1,0;y)+\\
&+4\,(-1+y)^2\,G(x\lambda,x_0,x\lambda;x,y)\,H(0,0;y)+\\
&+4\,(-1+y)\,G(x\lambda,x_1,\lambda;x,y)\,H(0,0;y)+
\end{aligned}
$$

$$+ 8\,(-1+y)\,G(x\lambda, x_1, \lambda; x, y)\,H(1, 0; y) +$$
$$+ 4\,(-1+y)^2\,G(x\lambda, x_1, x\lambda; x, y)\,H(0, 0; y) +$$
$$+ 4\,(-1+y)^2\,G(x\lambda, x\lambda, 0; x, y)\,H(0, 0; y) -$$
$$- 8\,(-1+y)^2\,G(x\lambda, x\lambda, 0; x, y)\,H(1, 0; x_0) +$$
$$+ 16\,G(\lambda, 0, 0, \lambda; x, y)\,H(0; x_0) - 8\,G(\lambda, 0, 0, \lambda; x, y)\,H(0; y) +$$
$$+ (8-8\,y)\,G(\lambda, 0, 0, x\lambda; x, y)\,H(0; y) + 24\,G(\lambda, 0, \lambda, 0; x, y)\,H(0; x_0) -$$
$$- 4\,G(\lambda, 0, \lambda, 0; x, y)\,H(0; y) + 8\,G(\lambda, 0, x_0, \lambda; x, y)\,H(0; x_0) -$$
$$- 4\,G(\lambda, 0, x_0, \lambda; x, y)\,H(0; y) + (4-4\,y)\,G(\lambda, 0, x_0, x\lambda; x, y)\,H(0; y) +$$
$$+ 8\,G(\lambda, 0, x_1, \lambda; x, y)\,H(0; x_0) - 4\,G(\lambda, 0, x_1, \lambda; x, y)\,H(0; y) +$$
$$+ (4-4\,y)\,G(\lambda, 0, x_1, x\lambda; x, y)\,H(0; y) + (4-4\,y)\,G(\lambda, 0, x\lambda, 0; x, y)\,H(0; y) +$$
$$+ 48\,G(\lambda, \lambda, 0, 0; x, y)\,H(0; x_0) - 8\,G(\lambda, \lambda, 0, 0; x, y)\,H(0; y) -$$
$$- 72\,G(\lambda, \lambda, \lambda, \lambda; x, y)\,H(0; x_0) + 36\,G(\lambda, \lambda, \lambda, \lambda; x, y)\,H(0; y) +$$
$$+ 36\,(-1+y)\,G(\lambda, \lambda, \lambda, x\lambda; x, y)\,H(0; y) + 8\,G(\lambda, x_0, 0, \lambda; x, y)\,H(0; x_0) -$$
$$- 4\,G(\lambda, x_0, 0, \lambda; x, y)\,H(0; y) + (4-4\,y)\,G(\lambda, x_0, 0, x\lambda; x, y)\,H(0; y) +$$
$$+ 16\,G(\lambda, x_0, \lambda, 0; x, y)\,H(0; x_0) - 8\,G(\lambda, x_0, \lambda, 0; x, y)\,H(0; y) +$$
$$+(4-4\,y)\,G(\lambda, x_0, x\lambda, 0; x, y)\,H(0; y) + 8\,G(\lambda, x_1, 0, \lambda; x, y)\,H(0; x_0) -$$
$$- 4\,G(\lambda, x_1, 0, \lambda; x, y)\,H(0; y) + (4-4\,y)\,G(\lambda, x_1, 0, x\lambda; x, y)\,H(0; y) +$$
$$+ 16\,G(\lambda, x_1, \lambda, 0; x, y)\,H(0; x_0) - 8\,G(\lambda, x_1, \lambda, 0; x, y)\,H(0; y) +$$
$$+ (4-4\,y)\,G(\lambda, x_1, x\lambda, 0; x, y)\,H(0; y) +$$
$$+ 24\,(-1+y)^2\,G(\lambda, x\lambda, x\lambda, \lambda; x, y)\,H(0; x_0) -$$
$$- 12\,(-1+y)^2\,G(\lambda, x\lambda, x\lambda, \lambda; x, y)\,H(0; y) -$$
$$- 12\,(-1+y)^3\,G(\lambda, x\lambda, x\lambda, x\lambda; x, y)\,H(0; y) -$$
$$- 8\,G(x_0, \lambda, 0, \lambda; x, y)\,H(0; x_0) + 4\,G(x_0, \lambda, 0, \lambda; x, y)\,H(0; y) +$$
$$+ 4\,(-1+y)\,G(x_0, \lambda, 0, x\lambda; x, y)\,H(0; y) - 16\,G(x_0, \lambda, \lambda, 0; x, y)\,H(0; x_0) +$$
$$+ 8\,G(x_0, \lambda, \lambda, 0; x, y)\,H(0; y) + 4\,(-1+y)\,G(x_0, \lambda, x\lambda, 0; x, y)\,H(0; y) +$$
$$+ 4\,(-1+y)\,G(x_0, x\lambda, \lambda, 0; x, y)\,H(0; y) - 8\,G(x_1, \lambda, 0, \lambda; x, y)\,H(0; x_0) +$$
$$+ 4\,G(x_1, \lambda, 0, \lambda; x, y)\,H(0; y) + 4\,(-1+y)\,G(x_1, \lambda, 0, x\lambda; x, y)\,H(0; y) -$$
$$- 16\,G(x_1, \lambda, \lambda, 0; x, y)\,H(0; x_0) + 8\,G(x_1, \lambda, \lambda, 0; x, y)\,H(0; y) +$$
$$+ 4\,(-1+y)\,G(x_1, \lambda, x\lambda, 0; x, y)\,H(0; y) + 4\,(-1+y)\,G(x_1, x\lambda, \lambda, 0; x, y)\,H(0; y) +$$
$$+ (8-8\,y)\,G(x\lambda, 0, \lambda, 0; x, y)\,H(0; y) + 24(-1+y)^2 G(x\lambda, \lambda, x\lambda, \lambda; x, y)\,H(0; x_0) -$$
$$- 12\,(-1+y)^2\,G(x\lambda, \lambda, x\lambda, \lambda; x, y)\,H(0; y)$$
$$- 12\,(-1+y)^3\,G(x\lambda, \lambda, x\lambda, x\lambda; x, y)\,H(0; y) +$$
$$+ (4-4\,y)\,G(x\lambda, x_0, \lambda, 0; x, y)\,H(0; y) + (4-4\,y)\,G(x\lambda, x_1, \lambda, 0; x, y)\,H(0; y) +$$
$$+24\,(-1+y)^2\,G(x\lambda, x\lambda, \lambda, \lambda; x, y)\,H(0; x_0) -$$
$$- 12\,(-1+y)^2\,G(x\lambda, x\lambda, \lambda, \lambda; x, y)\,H(0; y) -$$

$$
\begin{aligned}
&-12\,(-1+y)^3\,G(x\lambda,x\lambda,\lambda,x\lambda;x,y)\,H(0;y)+\\
&+4\,\Delta G(0;0,x_0)\,G(\lambda,0;0,x_0)\,G(\lambda,\lambda;x,y)-\\
&-4\,(-1+y)^2\,\Delta G(0;0,x_0)\,G(\lambda,0;0,x_0)\,G(x\lambda,x\lambda;x,y)+\\
&+4\,\Delta G(0;0,x_0)\,G(\lambda,\lambda;x,y)\,H(0,0;x_0)-\\
&-4\,\Delta G(0;0,x_0)\,G(\lambda,\lambda;x,y)\,H(1,0;x_0)-\\
&-4\,(-1+y)^2\,\Delta G(0;0,x_0)\,G(x\lambda,x\lambda;x,y)\,H(0,0;x_0)+\\
&+4\,(-1+y)^2\,\Delta G(0;0,x_0)\,G(x\lambda,x\lambda;x,y)\,H(1,0;x_0)+\\
&+4\,\Delta G(0;0,x_0)\,G(\lambda,0,\lambda;x,y)\,H(0;x_0)+\\
&+4\,\Delta G(0;0,x_0)\,G(\lambda,\lambda,0;x,y)\,H(0;x_0)+8\,\Delta G(0;0,x_0)\,G(\lambda,x_0,\lambda;x,y)\,H(0;x_0)+\\
&+8\,\Delta G(0;0,x_0)\,G(\lambda,x_1,\lambda;x,y)\,H(0;x_0)-\\
&-8\,\Delta G(0;0,x_0)\,G(x_0,\lambda,\lambda;x,y)\,H(0;x_0)-\\
&-8\,\Delta G(0;0,x_0)\,G(x_1,\lambda,\lambda;x,y)\,H(0;x_0)-\\
&-4\,(-1+y)^2\,\Delta G(0;0,x_0)\,G(x\lambda,x\lambda,0;x,y)\,H(0;x_0)+\\
&+8\,\Delta G(0;0,y)\,G(\lambda,\lambda;x,y)\,H(0,0;y)+\\
&+12\,(-1+y)\,\Delta G(0;0,y)\,G(\lambda,x\lambda;x,y)\,H(0,0;y)+\\
&+(8-8\,y)\,\Delta G(0;0,y)\,G(\lambda,x\lambda;x,y)\,H(1,0;y)+\\
&+12\,(-1+y)\,\Delta G(0;0,y)\,G(x\lambda,\lambda;x,y)\,H(0,0;y)+\\
&+(8-8\,y)\,\Delta G(0;0,y)\,G(x\lambda,\lambda;x,y)\,H(1,0;y)+\\
&+4\,(-1+y)\,\Delta G(0;0,y)\,G(\lambda,0,x\lambda;x,y)\,H(0;y)+\\
&+4\,(-1+y)\,\Delta G(0;0,y)\,G(\lambda,x_0,x\lambda;x,y)\,H(0;y)+\\
&+4\,(-1+y)\,\Delta G(0;0,y)\,G(\lambda,x_1,x\lambda;x,y)\,H(0;y)+\\
&+4\,(-1+y)\,\Delta G(0;0,y)\,G(\lambda,x\lambda,0;x,y)\,H(0;y)+\\
&+(4-4\,y)\,\Delta G(0;0,y)\,G(x_0,\lambda,x\lambda;x,y)\,H(0;y)+\\
&+(4-4\,y)\,\Delta G(0;0,y)\,G(x_0,x\lambda,\lambda;x,y)\,H(0;y)+\\
&+(4-4\,y)\,\Delta G(0;0,y)\,G(x_1,\lambda,x\lambda;x,y)\,H(0;y)+\\
&+(4-4\,y)\,\Delta G(0;0,y)\,G(x_1,x\lambda,\lambda;x,y)\,H(0;y)+\\
&+8\,(-1+y)\,\Delta G(0;0,y)\,G(x\lambda,0,\lambda;x,y)\,H(0;y)+\\
&+4\,(-1+y)\,\Delta G(0;0,y)\,G(x\lambda,\lambda,0;x,y)\,H(0;y)+\\
&+4\,(-1+y)\,\Delta G(0;0,y)\,G(x\lambda,x_0,\lambda;x,y)\,H(0;y)+\\
&+4\,(-1+y)\,\Delta G(0;0,y)\,G(x\lambda,x_1,\lambda;x,y)\,H(0;y)+\\
&+2\,\Delta G(0,0;0,x_0)\,G(\lambda,\lambda;x,y)\,H(0;x_0)-\\
&-2\,(-1+y)^2\,\Delta G(0,0;0,x_0)\,G(x\lambda,x\lambda;x,y)\,H(0;x_0)+\\
&+(4-4\,y)\,\Delta G(0,0;0,y)\,G(\lambda,x\lambda;x,y)\,H(0;y)+\\
&+(4-4\,y)\,\Delta G(0,0;0,y)\,G(x\lambda,\lambda;x,y)\,H(0;y)+\\
&+6\,\Delta G(0,\lambda;0,x_0)\,G(\lambda,\lambda;x,y)\,H(0;x_0)-
\end{aligned}
$$

$$
\begin{aligned}
&- 12\, \Delta G(0,\lambda;0,x_0)\, G(\lambda,\lambda;x,y)\, H(0;y) - \\
&- 6\, (-1+y)^2\, \Delta G(0,\lambda;0,x_0)\, G(x\lambda,x\lambda;x,y)\, H(0;x_0) + \\
&+ 12\, (-1+y)^2\, \Delta G(0,\lambda;0,x_0)\, G(x\lambda,x\lambda;x,y)\, H(0;y) + \\
&+ 8\, \Delta G(0,\lambda;0,y)\, G(\lambda,\lambda;x,y)\, H(0;y) - \\
&- 16\, (-1+y)\, \Delta G(0,\lambda;0,y)\, G(\lambda,x\lambda;x,y)\, H(0;x_0) + \\
&+ 8\, (-1+y)\, \Delta G(0,\lambda;0,y)\, G(\lambda,x\lambda;x,y)\, H(0;y) - \\
&- 16\, (-1+y)\, \Delta G(0,\lambda;0,y)\, G(x\lambda,\lambda;x,y)\, H(0;x_0) + \\
&+ 8\, (-1+y)\, \Delta G(0,\lambda;0,y)\, G(x\lambda,\lambda;x,y)\, H(0;y) + \\
&+ 6\, (-2\sqrt{y}+y)\, \Delta G(0,x\lambda;0,x_0)\, G(\lambda,\lambda;x,y)\, H(0;x_0) - \\
&- 6\, (-2+\sqrt{y})\, (-1+y)^2 \sqrt{y}\, \Delta G(0,x\lambda;0,x_0)\, G(x\lambda,x\lambda;x,y)\, H(0;x_0) + \\
&+ 4\, (-1+y)^2\, \Delta G(0,x\lambda;0,y)\, G(\lambda,x\lambda;x,y)\, H(0;y) + \\
&+ 4\, (-1+y)^2\, \Delta G(0,x\lambda;0,y)\, G(x\lambda,\lambda;x,y)\, H(0;y) + \\
&+ 4\, G(\lambda;0,x_0)\, G(\lambda,0;0,x_0)\, G(\lambda,\lambda;x,y) - \\
&- 4\, (-1+y)^2\, G(\lambda;0,x_0)\, G(\lambda,0;0,x_0)\, G(x\lambda,x\lambda;x,y) + \\
&+ 4\, G(\lambda;0,x_0)\, G(\lambda,\lambda;x,y)\, H(0,0;x_0) - 16\, G(\lambda;0,x_0)\, G(\lambda,\lambda;x,y)\, H(0,0;y) - \\
&- 4\, G(\lambda;0,x_0)\, G(\lambda,\lambda;x,y)\, H(1,0;x_0) \\
&- 4\, (-1+y)^2\, G(\lambda;0,x_0)\, G(x\lambda,x\lambda;x,y)\, H(0,0;x_0) + \\
&+ 16\, (-1+y)^2\, G(\lambda;0,x_0)\, G(x\lambda,x\lambda;x,y)\, H(0,0;y) + \\
&+ 4\, (-1+y)^2\, G(\lambda;0,x_0)\, G(x\lambda,x\lambda;x,y)\, H(1,0;x_0) + \\
&+ 4\, G(\lambda;0,x_0)\, G(\lambda,0,\lambda;x,y)\, H(0;x_0) - 8\, G(\lambda;0,x_0)\, G(\lambda,0,\lambda;x,y)\, H(0;y) + \\
&+ 4\, G(\lambda;0,x_0)\, G(\lambda,\lambda,0;x,y)\, H(0;x_0) - 8\, G(\lambda;0,x_0)\, G(\lambda,\lambda,0;x,y)\, H(0;y) + \\
&+ 8\, G(\lambda;0,x_0)\, G(\lambda,x_0,\lambda;x,y)\, H(0;x_0) - 16\, G(\lambda;0,x_0)\, G(\lambda,x_0,\lambda;x,y)\, H(0;y) + \\
&+ 8\, G(\lambda;0,x_0)\, G(\lambda,x_1,\lambda;x,y)\, H(0;x_0) - 16\, G(\lambda;0,x_0)\, G(\lambda,x_1,\lambda;x,y)\, H(0;y) - \\
&- 8\, G(\lambda;0,x_0)\, G(x_0,\lambda,\lambda;x,y)\, H(0;x_0) + 16\, G(\lambda;0,x_0)\, G(x_0,\lambda,\lambda;x,y)\, H(0;y) - \\
&- 8\, G(\lambda;0,x_0)\, G(x_1,\lambda,\lambda;x,y)\, H(0;x_0) + 16\, G(\lambda;0,x_0)\, G(x_1,\lambda,\lambda;x,y)\, H(0;y) - \\
&- 4\, (-1+y)^2\, G(\lambda;0,x_0)\, G(x\lambda,x\lambda,0;x,y)\, H(0;x_0) + \\
&+ 8\, (-1+y)^2\, G(\lambda;0,x_0)\, G(x\lambda,x\lambda,0;x,y)\, H(0;y) + \\
&+ 8\, G(\lambda;0,y)\, G(\lambda,\lambda;x,y)\, H(0,0;y) + 16\, G(\lambda;0,y)\, G(\lambda,\lambda;x,y)\, H(1,0;y) - \\
&- 32\, (-1+y)\, G(\lambda;0,y)\, G(\lambda,x\lambda;x,y)\, H(0,0;x_0) + \\
&+ 12\, (-1+y)\, G(\lambda;0,y)\, G(\lambda,x\lambda;x,y)\, H(0,0;y) - \\
&- 32\, (-1+y)\, G(\lambda;0,y)\, G(x\lambda,\lambda;x,y)\, H(0,0;x_0) + \\
&+ 12\, (-1+y)\, G(\lambda;0,y)\, G(x\lambda,\lambda;x,y)\, H(0,0;y) + \\
&+ (8-8\,y)\, G(\lambda;0,y)\, G(\lambda,0,x\lambda;x,y)\, H(0;x_0) + \\
&+ 4\, (-1+y)\, G(\lambda;0,y)\, G(\lambda,0,x\lambda;x,y)\, H(0;y) + \\
&+ (8-8\,y)\, G(\lambda;0,y)\, G(\lambda,x_0,x\lambda;x,y)\, H(0;x_0) +
\end{aligned}
$$

$$
\begin{aligned}
&+4\,(-1+y)\;G(\lambda;0,y)\,G(\lambda,x_0,x\lambda;x,y)\,H(0;y)+\\
&+(8-8\,y)\,G(\lambda;0,y)\,G(\lambda,x_1,x\lambda;x,y)\,H(0;x_0)+\\
&+4\,(-1+y)\;G(\lambda;0,y)\,G(\lambda,x_1,x\lambda;x,y)\,H(0;y)+\\
&+(8-8\,y)\,G(\lambda;0,y)\,G(\lambda,x\lambda,0;x,y)\,H(0;x_0)+\\
&+4\,(-1+y)\;G(\lambda;0,y)\,G(\lambda,x\lambda,0;x,y)\,H(0;y)+\\
&+8\,(-1+y)\;G(\lambda;0,y)\,G(x_0,\lambda,x\lambda;x,y)\,H(0;x_0)+\\
&+(4-4\,y)\,G(\lambda;0,y)\,G(x_0,\lambda,x\lambda;x,y)\,H(0;y)+\\
&+8\,(-1+y)\;G(\lambda;0,y)\,G(x_0,x\lambda,\lambda;x,y)\,H(0;x_0)+\\
&+(4-4\,y)\,G(\lambda;0,y)\,G(x_0,x\lambda,\lambda;x,y)\,H(0;y)+\\
&+8\,(-1+y)\;G(\lambda;0,y)\,G(x_1,\lambda,x\lambda;x,y)\,H(0;x_0)+\\
&+(4-4\,y)\,G(\lambda;0,y)\,G(x_1,\lambda,x\lambda;x,y)\,H(0;y)+\\
&+8\,(-1+y)\;G(\lambda;0,y)\,G(x_1,x\lambda,\lambda;x,y)\,H(0;x_0)+\\
&+(4-4\,y)\,G(\lambda;0,y)\,G(x_1,x\lambda,\lambda;x,y)\,H(0;y)-\\
&-16\,(-1+y)\;G(\lambda;0,y)\,G(x\lambda,0,\lambda;x,y)\,H(0;x_0)+\\
&+8\,(-1+y)\;G(\lambda;0,y)\,G(x\lambda,0,\lambda;x,y)\,H(0;y)+\\
&+(8-8\,y)\,G(\lambda;0,y)\,G(x\lambda,\lambda,0;x,y)\,H(0;x_0)+\\
&+4\,(-1+y)\;G(\lambda;0,y)\,G(x\lambda,\lambda,0;x,y)\,H(0;y)+\\
&+(8-8\,y)\,G(\lambda;0,y)\,G(x\lambda,x_0,\lambda;x,y)\,H(0;x_0)+\\
&+4\,(-1+y)\;G(\lambda;0,y)\,G(x\lambda,x_0,\lambda;x,y)\,H(0;y)+\\
&+(8-8\,y)\,G(\lambda;0,y)\,G(x\lambda,x_1,\lambda;x,y)\,H(0;x_0)+\\
&+4\,(-1+y)\;G(\lambda;0,y)\,G(x\lambda,x_1,\lambda;x,y)\,H(0;y)+\\
&+2\,G(\lambda,0;0,x_0)\,G(\lambda,\lambda;x,y)\,H(0;x_0)-\\
&-16\,G(\lambda,0;0,x_0)\,G(\lambda,\lambda;x,y)\,H(0;y)-\\
&-2\,(-1+y)^2\,G(\lambda,0;0,x_0)\,G(x\lambda,x\lambda;x,y)\,H(0;x_0)+\\
&+16\,(-1+y)^2\,G(\lambda,0;0,x_0)\,G(x\lambda,x\lambda;x,y)\,H(0;y)-\\
&-24\,(-1+y)\;G(\lambda,0;0,y)\,G(\lambda,x\lambda;x,y)\,H(0;x_0)+\\
&+(4-4\,y)\,G(\lambda,0;0,y)\,G(\lambda,x\lambda;x,y)\,H(0;y)-\\
&-24\,(-1+y)\;G(\lambda,0;0,y)\,G(x\lambda,\lambda;x,y)\,H(0;x_0)+\\
&+(4-4\,y)\,G(\lambda,0;0,y)\,G(x\lambda,\lambda;x,y)\,H(0;y)+\\
&+6\,G(\lambda,\lambda;0,x_0)\,G(\lambda,\lambda;x,y)\,H(0;x_0)-\\
&-12\,G(\lambda,\lambda;0,x_0)\,G(\lambda,\lambda;x,y)\,H(0;y)-\\
&-6\,(-1+y)^2\,G(\lambda,\lambda;0,x_0)\,G(x\lambda,x\lambda;x,y)\,H(0;x_0)+\\
&+12\,(-1+y)^2\,G(\lambda,\lambda;0,x_0)\,G(x\lambda,x\lambda;x,y)\,H(0;y)+\\
&+16\,G(\lambda,\lambda;0,y)\,G(\lambda,\lambda;x,y)\,H(0;x_0)-\\
&-8\,G(\lambda,\lambda;0,y)\,G(\lambda,\lambda;x,y)\,H(0;y)+
\end{aligned}
$$

$$+\,6\,\left(-2\,\sqrt{y}+y\right)\,G(\lambda,\lambda;x,y)\,G(\lambda,x\lambda;0,x_0)\,H(0;x_0)\,-$$
$$-\,16\,\left(-1+y\right)\,G(\lambda,\lambda;x,y)\,G(\lambda,x\lambda;0,y)\,H(0;y)\,+$$
$$+\,4\,G(\lambda,\lambda;x,y)\,G(x_0,\lambda;0,x_0)\,H(0;x_0)-8\,G(\lambda,\lambda;x,y)\,G(x_0,\lambda;0,x_0)\,H(0;y)\,+$$
$$+\,4\,\left(-2\,\sqrt{y}+y\right)\,G(\lambda,\lambda;x,y)\,G(x_0,x\lambda;0,x_0)\,H(0;x_0)\,+$$
$$+\,4\,G(\lambda,\lambda;x,y)\,G(x_1,\lambda;0,x_0)\,H(0;x_0)-8\,G(\lambda,\lambda;x,y)\,G(x_1,\lambda;0,x_0)\,H(0;y)\,+$$
$$+\,4\,\left(-2\,\sqrt{y}+y\right)\,G(\lambda,\lambda;x,y)\,G(x_1,x\lambda;0,x_0)\,H(0;x_0)\,-$$
$$-\,4\,G(\lambda,\lambda;x,y)\,H(0;x_0)\,H(0,0;y)-4\,G(\lambda,\lambda;x,y)\,H(0;y)\,H(0,0;x_0)\,-$$
$$-\,6\,\left(-2+\sqrt{y}\right)\,\left(-1+y\right)^2\,\sqrt{y}\,G(\lambda,x\lambda;0,x_0)\,G(x\lambda,x\lambda;x,y)\,H(0;x_0)\,+$$
$$+8\,(-1+y)^2\,G(\lambda,x\lambda;0,y)\,G(\lambda,x\lambda;x,y)\,H(0;x_0)\,-$$
$$-\,4\,(-1+y)^2\,G(\lambda,x\lambda;0,y)\,G(\lambda,x\lambda;x,y)\,H(0;y)\,+$$
$$+\,8\,(-1+y)^2\,G(\lambda,x\lambda;0,y)\,G(x\lambda,\lambda;x,y)\,H(0;x_0)\,-$$
$$-\,4\,(-1+y)^2\,G(\lambda,x\lambda;0,y)\,G(x\lambda,\lambda;x,y)\,H(0;y)\,-$$
$$-\,16\,\left(-1+y\right)\,G(\lambda,x\lambda;x,y)\,G(x_0,\lambda;0,y)\,H(0;x_0)\,+$$
$$+\,8\,\left(-1+y\right)\,G(\lambda,x\lambda;x,y)\,G(x_0,\lambda;0,y)\,H(0;y)\,+$$
$$+\,8\,(-1+y)^2\,G(\lambda,x\lambda;x,y)\,G(x_0,x\lambda;0,y)\,H(0;y)\,-$$
$$-\,16\,\left(-1+y\right)\,G(\lambda,x\lambda;x,y)\,G(x_1,\lambda;0,y)\,H(0;x_0)\,+$$
$$+\,8\,\left(-1+y\right)\,G(\lambda,x\lambda;x,y)\,G(x_1,\lambda;0,y)\,H(0;y)\,+$$
$$+\,8\,(-1+y)^2\,G(\lambda,x\lambda;x,y)\,G(x_1,x\lambda;0,y)\,H(0;y)\,+$$
$$+\,4\,\left(-1+y\right)\,G(\lambda,x\lambda;x,y)\,H(0;x_0)\,H(0,0;y)\,+$$
$$+\,4\,\left(-1+y\right)\,G(\lambda,x\lambda;x,y)\,H(0;y)\,H(0,0;x_0)\,-$$
$$-\,4\,(-1+y)^2\,G(x_0,\lambda;0,x_0)\,G(x\lambda,x\lambda;x,y)\,H(0;x_0)\,+$$
$$+\,8\,(-1+y)^2\,G(x_0,\lambda;0,x_0)\,G(x\lambda,x\lambda;x,y)\,H(0;y)\,-$$
$$-\,16\,\left(-1+y\right)\,G(x_0,\lambda;0,y)\,G(x\lambda,\lambda;x,y)\,H(0;x_0)\,+$$
$$+\,8\,\left(-1+y\right)\,G(x_0,\lambda;0,y)\,G(x\lambda,\lambda;x,y)\,H(0;y)\,-$$
$$-\,4\,\left(-2+\sqrt{y}\right)\,\left(-1+y\right)^2\,\sqrt{y}\,G(x_0,x\lambda;0,x_0)\,G(x\lambda,x\lambda;x,y)\,H(0;x_0)\,+$$
$$+8\,(-1+y)^2\,G(x_0,x\lambda;0,y)\,G(x\lambda,\lambda;x,y)\,H(0;y)\,-$$
$$-\,4\,(-1+y)^2\,G(x_1,\lambda;0,x_0)\,G(x\lambda,x\lambda;x,y)\,H(0;x_0)\,+$$
$$+\,8\,(-1+y)^2\,G(x_1,\lambda;0,x_0)\,G(x\lambda,x\lambda;x,y)\,H(0;y)\,-$$
$$-\,16\,\left(-1+y\right)\,G(x_1,\lambda;0,y)\,G(x\lambda,\lambda;x,y)\,H(0;x_0)\,+$$
$$+\,8\,\left(-1+y\right)\,G(x_1,\lambda;0,y)\,G(x\lambda,\lambda;x,y)\,H(0;y)\,-$$
$$-\,4\,\left(-2+\sqrt{y}\right)\,\left(-1+y\right)^2\,\sqrt{y}\,G(x_1,x\lambda;0,x_0)\,G(x\lambda,x\lambda;x,y)\,H(0;x_0)\,+$$
$$+8\,(-1+y)^2\,G(x_1,x\lambda;0,y)\,G(x\lambda,\lambda;x,y)\,H(0;y)\,+$$
$$+\,4\,\left(-1+y\right)\,G(x\lambda,\lambda;x,y)\,H(0;x_0)\,H(0,0;y)\,+$$
$$+\,4\,\left(-1+y\right)\,G(x\lambda,\lambda;x,y)\,H(0;y)\,H(0,0;x_0)\,-$$
$$-\,4\,(-1+y)^2\,G(x\lambda,x\lambda;x,y)\,H(0;x_0)\,H(0,0;y)\,-$$

$$
\begin{aligned}
&-4\,(-1+y)^2\,G(x\lambda,x\lambda;x,y)\,H(0;y)\,H(0,0;x_0)-4\,G(\lambda,\lambda,0;x,y)\,H(0;x_0)\,H(0;y)+\\
&+4\,(-1+y)\;G(\lambda,x\lambda,0;x,y)\,H(0;x_0)\,H(0;y)+\\
&+4\,(-1+y)\;G(x\lambda,\lambda,0;x,y)\,H(0;x_0)\,H(0;y)-\\
&-4\,(-1+y)^2\,G(x\lambda,x\lambda,0;x,y)\,H(0;x_0)\,H(0;y)\,.
\end{aligned}
$$

As with the other MI, we have checked that the leading (finite) contribution agrees with the results of ref. [54].

7. Summary

In this paper, we have provided series expansions in the dimensional regularisation parameter ϵ for all two-loop master integrals with three external off-shell legs and all internal lines being massless. The results are presented in terms of an extended basis of 2-dimensional harmonic polylogarithms. The novel feature is that this basis includes quadratic forms - that matches on to the allowed phase space boundary for the $1 \to 2$ decay. For each master integral, we have given sufficient terms in the ϵ-expansion to describe two-loop vertex corrections for physical processes.

The MI presented here are ingredients for a variety of interesting two-loop processes such as the QCD corrections to $H \to V^*V^*$ decay in the heavy top quark limit and the QCD corrections to the fully off-shell triple gluon (and quark-gluon) vertices.

The MI also form a staging post for the study of massless two-loop $2 \to 2$ scattering amplitudes with two off-shell legs. These processes include the NNLO QCD corrections to $q\bar{q} \to V^*V^*$ (where $V = W$,Z) and the NLO corrections to $gg \to V^*V^*$. Altogether there are 11 planar box and 3 non-planar box master topologies which remain to be studied.

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